ANALYTICAL STATICS.

A SUPPLEMENT

TO THE

FOURTH EDITION OF AN ELEMENTARY TREATISE

ON

MECHANICS.

By W. WHEWELL, M.A. F.R.S.

FELLOW AND TUTOR OF TRINITY COLLEGE.

CAMBRIDGE:

PRINTED AT THE PITT PRESS, BY JOHN SMITH,
PRINTER TO THE UNIVERSITY.

FOR J. & J. J. DEIGHTON, TRINITY STREET, CAMBRIDGE;
AND WHITTAKER & ARNOT, LONDON.

M.DCCC.XXXIII.

PREFACE.

Having thought it advisable, in the fourth edition of my "Elementary Treatise on Statics," to separate from the absolutely elementary portion, those parts which assume the Student to possess a knowledge of Analytical Geometry and of the Differential Calculus, I have ventured to affix to this separate publication the title of Analytical Statics. But this title is to be understood rather as indicating the nature of the subject, than as promising a complete Treatise upon it. Such a Treatise would imply an extension of the plan of the former work for which I was not prepared when a new edition was called for. I have, however, inserted a few of the deficient propositions which seem most important in such a work; for instance, an independent proof of the Composition of Forces acting at a point, (Chap. I.) and a proof of the general principle of Virtual Velocities (Art. 20—22). In addition to these, there are other propositions, remarkable for their Analytical generality or beauty, which might properly form parts of a Treatise on Statics: for instance;—the Theory of Moments;—and certain propositions founded upon the principle of Virtual Velocities. For these subjects I may refer to Mr Poisson's Treatise, Articles 271 to 284, and 346 to 349, of the Second Edition.

I have, for reasons already stated, been desirous of introducing, as far as can conveniently be done, propositions which have a bearing upon practical applications of mechanical knowledge; and have, with this view, bor-

rowed from the excellent Memoirs of Mr Davies Gilbert and Mr Hodgkinson, on Suspension Bridges, some of the most important portions. I have also introduced a Chapter upon the Strength of Materials with regard to Fracture. It is very obviously desirable that this subject should enter into our Treatises of Mechanics; but the difficulty of effecting this may be supposed to be considerable, when we perceive that the theory of Galileo, which assumes that materials are absolutely incapable of being either condensed or broken by compression, has held its place in some of our most received Treatises up to the present time; notwithstanding its complete repugnance both to our general conceptions of the structure of materials, and to the results of observation. The researches of Mr Barlow and others, and still more recently the well devised experiments and clear views of Mr Hodgkinson, have, it may be hoped, done much to put us in possession of a theory of this subject, consistent with itself and with facts. In this hope I have endeavoured to bring the subject before the Student of Mechanics, following principally the investigations of Mr Hodgkinson, as contained in the Transactions of the Manchester Society. Though discrepancies and difficulties may still exist with regard to this matter, they will probably disappear in the course of further researches, if the fundamental principles are rightly established.

I have omitted the investigations concerning the Forms of Bridges on various hypotheses, and the discussion of the Species of the Elastic Curve, which made part of the former Editions. These portions of the work are not suited to the Mathematical Student as parts of a Course of Mechanics, and occupied too much space for mere examples.

CONTENTS.

ERRATA.

PAGE	LINE	ERROR.	CORRECTION.
4	18	$2r, \phi(\theta)$	$2r\,\phi(\theta)$
5	19	whence	when
66	15	z	x
103	12 in the heading of the Table	C	c.

ANALYTICAL STATICS.

CHAP. I.

RESOLUTION OF A FORCE INTO RECTANGULAR COMPONENTS.

1. Statics is that part of the science of Mechanics which treats of forces employed in producing equilibrium.

The forces treated of in this part of Mechanics are *pressures*. They are measured by the number of units of pressure to which they are equivalent; each unit of pressure being supposed capable of producing an equal effect in maintaining equilibrium.

When two or more forces act at the same point in any directions, they produce an effect which may be produced by a single force acting at the same point. The two forces are said to be *compounded* into the single force; the two forces are called the *components;* the single force is called the *resultant*. If the single force be the one first considered, it is said to be *resolved* into the two component forces.

When two forces act in the same direction, their resultant is their sum; when they act in opposite directions, their resultant is their difference, and is in the direction of the greater.

If a point be acted upon by any two forces, and also by a force equal and opposite to their resultant, it will be kept in equilibrium: for this is equivalent to its being acted on by the resultant and by a force equal and opposite to the resultant.

2. The relations of forces which keep each other in equilibrium may be deduced, beginning either with the con-

sideration of forces acting at a point, or of forces acting on a lever. Each case may be deduced from the other. In the Elementary Treatise, to which the present volume is a Supplement, the latter plan was followed. It was there shewn (Art. 27), that if two forces p, q, act at any angle, their resultant r is represented, in magnitude and direction, by the diagonal of a parallelogram of which the sides similarly represent p, q. It hence follows, that if p, q act at right angles to each other, and if θ be the angle which r makes with p, we have

$$p = r\cos\theta, \quad q = r\sin\theta, \quad \tan\theta = \frac{q}{p}, \quad r = \sqrt{(p^2 + q^2)}.$$

By means of these expressions we may treat all the problems of Statics analytically; and this we shall proceed to do for some of the most important.

But, in order to make the analytical mode of treating the subject more complete, we shall first prove the above expressions independently, beginning with the consideration of forces acting at a point.

3. PROP. *If a force* r *be resolved into two forces* p, q, *at right angles to each other, and if* θ *be the angle between* r *and* p, *the ratio* $\frac{\mathrm{p}}{\mathrm{r}}$ *is the same so long as* θ *is the same.*

Let n equal forces r', r', &c. act in the direction of r; and let p', q' be the components of each in the directions of p, q; the components of each of the equal forces r', r', &c. will be equal. Hence, nr' will be the force in the direction of r, and np', nq' the components, whatever n may be. And the angle θ is the angle between r' and p', and therefore between r and p. Therefore θ is the same so long as r and p are represented by nr' and np'; that is, so long as r and p are in the ratio of r' and p'.

Hence, θ is the same so long as $\frac{p}{r}$ is the same, and one of these quantities varies only when the other does. Therefore also $\frac{p}{r}$ is the same so long as θ is the same.

Cor. 1. We may express this dependence by saying, that $\frac{p}{r}$ is a *function* of θ; or in symbols

$$\frac{p}{r} = \phi(\theta),$$

$\phi(\theta)$ representing a function of θ hereafter to be determined.

Cor. 2. It is obvious that q will depend on $\frac{\pi}{2} - \theta$, in the same manner in which p depends on θ; hence

$$\frac{q}{r} = \phi\left(\frac{\pi}{2} - \theta\right).$$

We shall in what follows suppose p, q to be the rectangular components of r, and θ the angle between r and p; and shall express the dependence of p on r and θ in the manner just stated; whence we have $p = r\phi(\theta)$.

4. Prop. *If two forces, each equal to* r, *act at an angle* 2θ, *and produce a resultant* s, *we shall have*

$$s = 2r\phi(\theta).$$

Let AR, AR', fig. 91, be the directions in which the equal forces r, r' act; so that RAR' is 2θ. Bisect the angle RAR' by the line AS, and draw QAQ' perpendicular to AS.

Let the force r, which acts in AR, be resolved into p acting in the direction AS, and q acting in the direction AQ. Then by the last Article, since $RAS = \theta$,

$$p = r\phi(\theta), \quad q = r\phi\left(\frac{\pi}{2} - \theta\right).$$

In like manner, the force r' equal to r, which acts in AR' may be resolved into p' acting in AS and q' acting in AQ'; and, as before,

$$p' = r\phi(\theta), \quad q' = r\phi\left(\frac{\pi}{2} - \theta\right).$$

The two forces, q, q', are equal and in opposite directions, and therefore destroy each other; and therefore the resultant

of the forces r, r', is the resultant of p and p', which, since they are in the same direction, is the sum of p and p', or $2r\phi(\theta)$. Therefore the resultant $s = 2r\phi(\theta)$.

5. **Prop.** *If, for any given angle 2θ, we have $\phi(2\theta) = \cos. 2\theta$, we shall also have $\phi(\theta) = \cos. \theta$.*

Let, in fig. 92, two equal forces q, q' act at an angle QAQ' which is 4θ; the resultant will be s, in the direction AS which bisects PAP', and we shall have $s = 2q\phi(2\theta)$.

Suppose that besides the two forces q, q', in AQ, AQ', two forces p, p', equal to these, act in AP. Then the resultant of the four forces p, p', q, q' will be $2p + 2q\phi(2\theta)$.

But the two equal forces p in AP and q in AQ are equivalent to a force r in AR which bisects QAS. And in like manner the two forces p in AP and q in AQ' are equivalent to a force r in AR'. And by last Art. we have in each case

$$r = 2p\phi(\theta).$$

Again, the two forces r in AR and r in AR' are equivalent to a force $2r$, $\phi(\theta)$ by the same Article; that is, putting for r its value, the two forces r, r', or the four forces p, p', q, q', are equivalent to $4p\{\phi(\theta)\}^2$.

Hence we have, putting p for q

$$4p\{\phi(\theta)\}^2 = 2p + 2p\phi(2\theta);$$

$$\phi(\theta) = \sqrt{\frac{1 + \phi(2\theta)}{2}}.$$

By comparing this with the trigonometrical formula

$$\cos. \theta = \sqrt{\frac{1 + \cos. 2\theta}{2}},$$

it appears that if $\phi(2\theta)$ is $\cos. 2\theta$, $\phi\theta$ will be $\cos. \theta$.

Cor. It appears also that if $\phi(\theta)$ is $\cos. \theta$, $\phi(2\theta)$ will be $\cos. 2\theta$.

6. Prop. *For all angles which can be obtained by the continual bisection of an angle of* 60 *degrees,* $\phi(\theta)$ *is* cos. θ.

Let two equal forces q, q', act in AQ, AQ', fig. 92, at an angle of 120^0, so that θ is 60^0. Their resultant, s, in the direction AP, will bisect the angle QAQ', whence PAQ will be 60^0; and if PA be produced to O, QAO will be 120^0. If a force s' equal to s, the resultant of q, q', be applied at A in the direction AO, the three q, q', s', will keep the point A in equilibrium. But in this case, since the three angles QAQ', QAO, $Q'AO$ are equal, the three forces q, q', s' must be equal: for each may be considered as the resultant of the other two, and each pair act at the same angle. Therefore $s' = q$, and $s = q$.

Now by the last Article $s = 2q\phi(\theta)$; whence we have $q = 2q\phi(\theta)$ and $\phi(\theta) = \frac{1}{2}$, when θ is 60^0. And cos. $60^0 = \frac{1}{2}$; whence it appears that in this case $\phi(\theta)$ is cos. (θ).

Hence it follows, by the last Article, that $\phi(\theta)$ is cos. θ, whence θ is 30^0. Hence again $\phi(\theta)$ is cos. θ, when θ is 15^0; and so on, to any arc which can be obtained by the continual bisection of 60^0.

Cor. By continual bisection we may make the angle θ smaller than any assigned angle, and the proposition will still be true.

7. Prop. *If* $\phi(\theta)$ *be* cos. θ, *and* $\phi(\epsilon)$ *be* cos. ϵ, *and* $\phi(\theta-\epsilon)$ *be* cos. $(\theta-\epsilon)$; *then also* $\phi(\theta+\epsilon)$ *is* cos. $(\theta+\epsilon)$.

Let two equal forces r, r', in AR, AR', fig. 93, make an angle 2θ; and let s in AS be their resultant. Let r, which acts in AR, be the resultant of two equal forces p, q, acting in AP, AQ, at an angle 2ϵ. Therefore RAP will be ϵ, and $r = 2p\,\phi(\epsilon)$. Also SAR will be θ, and $s = 2r\,\phi(\theta)$

$$= 4p\,\phi(\epsilon)\,.\,\phi(\theta).$$

The force s is the resultant of the two forces p, p' acting in AP, AP', and the two q, q' acting in AQ, AQ'. And the two equal forces p, p', will have a resultant in AS, which

by Art. 4. will be $2p\,\phi\,(\theta-\epsilon)$, since PAS is $\theta-\epsilon$. Also the two q, q' will have a resultant $2q\,\phi\,(\theta+\epsilon)$ in AS, since QAS is $\theta+\epsilon$. Hence s, the resultant of the force p, p', q, q', will be $2p\,\phi\,(\theta-\epsilon)+2q\,\phi\,(\theta+\epsilon)$; and since q is equal to p, equating this with the former value of s, we have

$$2p\phi\,(\theta-\epsilon)+2p\phi\,(\theta+\epsilon)=4\phi\,(\epsilon)\,.\,\phi\,(\theta).$$

$$\text{Hence}\ \phi(\theta+\epsilon)=2\phi(\epsilon)\,.\,\phi(\theta)-\phi(\theta-\epsilon).$$

Now we have by trigonometry,

$$\cos.\,(\theta+\epsilon)=2\cos.\,\epsilon\,.\cos.\,\theta-\cos.\,(\theta-\epsilon).$$

If therefore

$\phi(\theta)$ be $\cos.\theta$, $\phi(\epsilon)$ be $\cos.\epsilon$, and $\phi(\theta-\epsilon)$ be $\cos.\,(\theta-\epsilon)$,

we shall have also $\phi(\theta+\epsilon)=\cos.\,(\theta+\epsilon)$.

8. Prop. *For all values of* θ, $\phi(\theta)$ *is* $\cos.\theta$.

Whatever be the value of θ, we may, by the perpetual bisection of an angle of 60^0, obtain an angle which either measures θ, or measures it with a remainder less than any assigned angle, since we may by the perpetual bisection of an angle of 60^0 obtain an angle less than any assigned angle. We will therefore suppose $\theta=n\delta$, when δ is an arc obtained by the perpetual bisection of an angle of 60^0, and n is a a whole number.

It is proved, Art. 6, that $\phi\,(\delta)$ is $\cos.\delta$. It hence follows by Cor. to Art. 5, that $\phi\,(2\delta)$ is $\cos.\,2\delta$.

Also since this is true for 2δ, δ, and $2\delta-\delta$, it is true, by Art. 7, for $2\delta+\delta$ or 3δ.

Again since it is true for 3δ, δ, and $3\delta-\delta$ or 2δ, it is true for $3\delta+\delta$, or 4δ.

Again since it is true for 4δ, δ, and $4\delta-\delta$ or 3δ, it is true for $4\delta+\delta$, or 5δ.

And in this manner it may be proved for $n\delta$, when n is any whole number.

Therefore, by what has been said, it is true for θ.

Hence if p, q be the rectangular components of a force r, of which p makes with r an angle θ,

$$p = r \cos. \theta.$$

Cor. 1. Hence also $q = r \cos. \left(\frac{\pi}{2} - \theta\right) = r \sin. \theta$.

Cor. 2. Hence $\tan. \theta = \dfrac{r \sin. \theta}{r \cos. \theta} = \dfrac{p}{q}$.

Cor. 3. Also $r^2 = r^2 \cos^2. \theta + r^2 \sin.^2 \theta = p^2 + q^2$.

CHAP. II.

THE CONDITIONS OF EQUILIBRIUM OF A POINT.

9. In this and the following Chapter we shall express the conditions which are requisite that a point or a body may be in equilibrium, by means of equations among the symbols which the forces and their positions introduce; and we shall thus obtain the means of reducing to the solution of equations, all problems whatever relative to equilibrium.

Prop. *To find the resultant of two forces acting at a point, as AP, AQ, fig. 94.*

If we suppose a line, as Ax, the position of which is known, to pass through the point A, we may determine the positions, both of the components, and of the resultant, by the angles which they make with this line.

Let p, q, be the forces in AP, AQ; α, β, the angles which they make with Ax. If p be resolved into two forces,

one in the direction Ax, and the other in the direction A perpendicular to Ax; it has been shewn, in the precedin Chapter, that these resolved parts will be $p \cos. \alpha$, $p \sin. \alpha$ In the same manner q is equivalent to forces $q \cos. \beta$ in th direction Ax, and $q \sin. \beta$ in the direction Ay. Hence th forces p, q are equivalent to

$$p \cos. \alpha, \ q \cos. \beta \text{ in } Ax,$$
$$p \sin. \alpha, \ q \sin. \beta \text{ in } Ay;$$

and the resultant of p and q will be the resultant of these four forces. If we put

$$p \cos. \alpha + q \cos. \beta = X,$$
$$p \sin. \alpha + q \sin. \beta = Y;$$

and if r be the resultant of p and q, and θ the angle which it makes with Ax, we have by Art. 8, Cor. 2, 3,

$$r = \sqrt{(X^2 + Y^2)}, \ \tan. \theta = \frac{Y}{X}.$$

whence the magnitude and position of the resultant are known.

COR. 1. By putting the values of X and Y in the expression for r, we find

$$r = \sqrt{\begin{Bmatrix} p^2 \cos.^2 \alpha + 2pq \cos. \alpha . \cos. \beta + q^2 \cos.^2 \beta \\ + p^2 \sin.^2 \alpha + 2pq \sin. \alpha . \sin. \beta + q^2 \sin.^2 \beta \end{Bmatrix}},$$

and since $\cos.^2 \alpha + \sin.^2 \alpha = 1$,

and $\cos. \alpha . \cos. \beta + \sin. \alpha . \sin. \beta = \cos. (\alpha - \beta)$,

$$r = \sqrt{\{p^2 + 2pq \cos. (\alpha - \beta) + q^2\}}.$$

COR. 2. This agrees with the result obtained in Chap. II. of the Elementary Treatise; for if AP, AQ represent the forces p, q, and if AR be found by completing the parallelogram $APRQ$, we shall have

$$AR^2 = AP^2 + PR^2 + 2AP . PR . \cos. RPE,$$
$$\text{or } = p^2 + q^2 + 2pq \cos. (\alpha - \beta),$$
$$\text{because } RPE = QAP = PAx - QAx.$$

Cor. 3. If we call the angles PAR and QAR, ϕ and ψ respectively, we shall have, by Trigonometry,

$$\frac{\sin. PAR}{\sin. APR} = \frac{PR}{AR}, \text{ or}$$

$$\frac{\sin. PAR}{\sin. EPR} = \frac{A}{AR};$$

$$\therefore \frac{\sin. \phi}{\sin. (\alpha - \beta)} = \frac{q}{r},$$

$$\sin. \phi = \frac{p \sin. (\alpha - \beta)}{r} = \frac{q \sin. (\alpha - \beta)}{\sqrt{\{p^2 + 2pq \cos. (\alpha - \beta) + q^2\}}}.$$

Similarly

$$\sin. \psi = \frac{p \sin. (\alpha - \beta)}{r} = \frac{p \sin. (\alpha - \beta)}{\sqrt{\{p^2 + 2pq \cos. (\alpha - \beta) + q^2\}}}.$$

10. Prop. *To find the resultant of any number of forces* $p_1, p_2, p_3 \dots p_n$ *in the same plane; their directions making with the line* Ax, *angles* $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ *respectively.*

As in the last Article, Ay being perpendicular to Ax, the forces may be shewn to be equivalent to

$p_1 \cos. \alpha_1, p_2 \cos. \alpha_2, p_3 \cos. \alpha_3 \dots p_n \cos. \alpha_n$ in the direction Ax,
$p_1 \sin. \alpha_1, p_2 \sin. \alpha_2, p_3 \sin. \alpha_3 \dots p_n \sin. \alpha_n$ in the direction Ay.

Hence, if r be the resultant, and θ the angle which it makes with Ax, r and θ will be given by the equations

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 \dots + p_n \cos. \alpha_n = X,$$
$$p_1 \sin. \alpha_1 + p_2 \sin. \alpha_2 + p_3 \sin. \alpha_3 \dots + p_n \sin. \alpha_n = Y,$$

$$r = \sqrt{(X^2 + Y^2)}; \tan. \theta = \frac{Y}{X}.$$

We have considered the forces as lying within the angle yAx and *pulling* the body. In this case the resolved parts will be in the directions Ax and Ay; but if one of the forces act in the direction AP', fig. 95, situated in the angle yAx',

the resolved part AM' will act in the direction xA, and the corresponding term is the sum $p_1 \cos. a_1 + p_2 \cos. a_2 +$ &c. should be negative. And if $p' \cos. a'$ be this term, it will be negative, because $a' = P'Ax$, and $p' \cos. a' = p' \cos. P'Ax = -p' \cos. P'Ax'$, which is a negative quantity. In this case $p' \sin. a'$ will be positive, which agrees with the direction of the resolved force $M'P'$.

In the same manner, if a force p'' act in the direction AP'', in the quadrant $y'Ax$, the term $p'' \sin. a''$ will be negative, and $p'' \cos. a''$ will be positive.

And if a force p''' act in the direction AP''' in the quadrant $y'Ax'$, the terms $p''' \cos. a'''$, $p''' \sin. a'''$, will both be negative. And these changes of sign agree with the changes of direction of the resolved parts.

And if the force, instead of being a *pulling* force in the direction AP, be a *pushing* force in the direction PA, we must make p negative; and the resolved parts $p \cos. a$ and $p \sin. a$ will both be negative. In the same manner if the force in $P'A$ be a pushing force, we must make p' negative. And similarly in the other quadrants.

11. Prop. *To find the resultant of forces whose directions are not all in the same plane.*

We have in the preceding case resolved forces in the directions of two lines at right angles to each other. In this case we shall resolve them in the directions of three lines, each at right angles to the other two. The nature of space admits of three such lines, or *axes*, and no more. Let xAy, fig. 96, be conceived to be a horizontal plane, in which Ax and Ay are at right angles; and let Az be vertical. Then Ax, Ay, Az are all at right angles to each other; and the planes xAy, xAz, yAz are also at right angles to each other. For (Euc. XI. Def. 6.), yAx measures the inclination of zAy, zAx. And similarly of the others.

Let P be any point in space; and through P let three planes be drawn, $PmOn$, $PoNm$, $PoMn$, parallel respectively

to xAy, xAz, yAz. Hence Mm will be a rectangular parallelopiped; and therefore the plane nMo is perpendicular to AMo, AMn. Therefore AM is perpendicular to the plane nMo (Euc. XI. 19.), and therefore to the line PM (Euc. XI. 4.).

If AP represent any force acting at A, AP may be resolved into forces represented by AM, MP. Also MP may be resolved into Mo, oP; and hence the force AP is equivalent to AM, Mo, oP; or to AM, AN, AO.

Since PM is perpendicular to AM, $AM = AP \,.\cos. PAx$. And similarly $AN = AP \,.\cos. PAy$ and $AO = AP \,.\cos. PAz$. Hence if p be the force AP, and α, β, γ, the angles which it makes with Ax, Ay, Az, the force will be equivalent to three forces

$$p \cos.\alpha \text{ in } Ax, \quad p \cos.\beta \text{ in } Ay, \quad p \cos.\gamma \text{ in } Az^{*}.$$

Hence if we have forces $p_1, p_2, p_3, \ldots p_n$, acting at a point A

$$\begin{aligned} \text{making with } Ax_1 \text{ angles } & \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n; \\ \text{with } Ay_1 \text{ angles } & \beta_1, \beta_2, \beta_3, \ldots \beta_n; \\ \text{with } Az_1 \text{ angles } & \gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n; \end{aligned}$$

and if we make

$$p_1 \cos.\alpha_1 + p_2 \cos.\alpha_2 + p_3 \cos.\alpha_3 \ldots + p_n \cos.\alpha_n = X;$$

$$p_1 \cos.\beta_1 + p_2 \cos.\beta_2 + p_3 \cos.\beta_3 \ldots + p_n \cos.\beta_n = Y;$$

$$p_1 \cos.\gamma_1 + p_2 \cos.\gamma_2 + p_3 \cos.\gamma_3 \ldots + p_n \cos.\gamma_n = Z;$$

the forces will be equivalent to X in Ax, Y in Ay, and Z in Az.

---

* *Two* of the angles α, β, γ are sufficient to determine the position of the line AP, for they are connected by the equation

$$\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1;$$

so that two of them being known, the third may be found.

This appears thus;

$$\begin{aligned} AP^2 &= AM^2 + MP^2 = AM^2 + Mo + oP^2 \\ &= AM^2 + AN^2 + PO^2 \\ &= AP^2 \cos.^2\alpha + AP^2 \cos.^2\beta + AP^2 \cos.^2\gamma; \\ \therefore 1 &= \cos.\alpha^2 + \cos.^2\beta + \cos.^2\gamma. \end{aligned}$$

If R be the resultant, and θ, η, ζ the angles which it makes with Ax, Ay, Az respectively, we shall have

$$R = \sqrt{(X^2 + Y^2 + Z^2)},$$

$$\cos. \theta = \frac{X}{R}, \cos. \eta = \frac{Y}{R}, \cos. \zeta = \frac{Z}{R}.$$

For if AM, AN, AO now represent X, Y, Z, AP will represent R; and $AP^2 = AM^2 + AN^2 + AO^2$ (see note last page).

$$\text{Also } AM = AP \cos. PAM, \text{ \&c.}$$

One of the three last equations is superfluous, as was observed before.

As in the last Article, the resolved forces may become negative when the angles α_1, β_1, γ_1, &c. pass beyond the first quadrant. Also the forces are negative when they push instead of pulling.

12. Prop. *When a point is acted upon by any forces, to find the conditions of equilibrium.*

In order that there may be an equilibrium, the resultant of all the forces must be 0. And in order that this may be the case it is evident that we must have, in Art. 10, $X = 0$, $Y = 0$; and, in Art. 11, $X = 0$, $Y = 0$, $Z = 0$. Hence we have for the conditions of equilibrium in the former case,

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \ldots = 0;$$

$$p_1 \sin. \alpha_1 + p_2 \sin. \alpha_2 + p_3 \sin. \alpha_3 + \ldots = 0.$$

And in the latter case

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \ldots = 0;$$

$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 + \ldots = 0;$$

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 + \ldots = 0.$$

The conditions of the equilibrium of any number of points may be deduced from the conditions belonging to one point. In the state of equilibrium, each point, by

means of the rods, strings, &c. which connect it with the other points, exerts and suffers a certain pressure. And this pressure may be introduced as one of the forces at each point, and then eliminated by considering that it is equal at each two points so connected.

13. PROP. *When a body is supported upon a curve (the curve being in a vertical plane); to find the conditions of equilibrium.*

Let AM, MP, fig. 97, be the vertical abscissa and the horizontal ordinate of the curve; and let $AM = x$, $MP = y$, $BP = s$. Let the forces which act on the body be resolved in the directions parallel to x and to y, and let the resolved parts thus obtained be called X and Y: X and Y being considered positive when they tend to increase x and y. Also let R be the re-action of the curve in the direction of the normal, or what is the same thing, the pressure of the body on the curve. Then, in order to obtain the conditions of equilibrium, resolve R in the directions parallel to x and to y;

$\therefore$ resolved part of R in direction $PX = R \cos. RPX$

$$= R \sin. XPT = R \cdot \frac{dy}{ds},$$

resolved part of R in direction $PY = R \cos. RPY$

$$= - R \cos. RPM = - R \cdot \frac{dx}{ds}.$$

Hence, by Art. 12, the equilibrium will subsist if

$$X + R\frac{dy}{ds} = 0;$$

$$Y - R\frac{dx}{ds} = 0.$$

The first of these equations is equivalent to

$$X + R\frac{dy}{dx}\frac{dx}{ds} = 0$$

and if we multiply the second by $\dfrac{dy}{dx}$ and add it to this, we have

$$X + Y\frac{dy}{dx} = 0,$$

which is the equation of equilibrium. If we know the curve, that is, the relation between x and y, this equation will give us the relation between X and Y; and if we know this also, the equation will enable us to find the actual values of x and y, or the point when the body will be supported. This will be illustrated by the problems which follow.

If the weight P, instead of resting upon a material surface BP, fig. 97, be suspended by a string KP which confines it to the curve BP, the conditions of equilibrium will be the same as before. The re-action which was before supplied by the resistance of the surface is now produced by the tension of the string. This re-action will as before be perpendicular to the curve: it will also manifestly be in the direction of the string, and this agrees with what is collected from the way in which the curve is described; for when a curve is traced out by one end of a string of which the other is fixed, the string will at every point be perpendicular to the curve Hence the formulæ which we are about to give for the former case apply also to this.

14. Prop. *A body is supported upon a curve by a weight acting over a fixed pully K, fig.* 98; *to find the conditions of equilibrium.*

Take the vertical line KM, passing through the pully, for the line on which x is measured downwards.

Let $KM = x$, $MP = y$, $KP = r = (x^2 + y^2)^{\frac{1}{2}}$; and if the weight which acts by means of KP be $= q$, the parts which act parallel to MK and PM are

$$q\frac{x}{r}, \text{ and } q\frac{y}{r};$$

$$\text{hence } X = p - q\frac{x}{r}; \quad Y = -q\frac{y}{r};$$

hence the equation of Art. 13, namely, $X + Y\dfrac{dy}{dx} = 0$, becomes

$$p - q\frac{x}{r} - q\frac{ydy}{rdx} = 0,$$

$$\text{or } p - q\left(\frac{x}{r} + \frac{ydy}{rdx}\right) = 0;$$

$$\text{but } x^2 + y^2 = r^2; \ \therefore \ x + \frac{ydy}{dx} = \frac{rdr}{dx},$$

$$\text{and } \frac{x}{r} + \frac{ydy}{rdx} = \frac{dr}{dx};$$

$$\therefore \ p - q\frac{dr}{dx} = 0,$$

and this, combined with the relation between x and r, which is given by the nature of the curve, gives the position of equilibrium.

15. Prob. I. *Let AP, fig. 98, be a hyperbola with its axis vertical, on which a given weight P is supported by another given weight Q by means of a string passing over a pully at the center; to find the position of equilibrium.*

Let as before KM, MP, AP, be x, y, r; the semi-axes of the hyperbola a and b: the given weights p and q.

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2);$$

$$\therefore \ r = (x^2 + y^2)^{\frac{1}{2}} = \left(x^2 + \frac{b^2x^2}{a^2} - b^2\right)$$

$$= \left(\frac{a^2 + b^2}{a^2}x^2 - b^2\right)^{\frac{1}{2}}$$

$$= (e^2x^2 - b^2)^{\frac{1}{2}}:$$

making $e = \dfrac{(a^2 + b^2)^{\frac{1}{2}}}{a}$, which is called the *eccentricity* of the hyperbola;

$$\therefore \frac{dr}{dx} = \frac{e^2 x}{(e^2 x^2 - b^2)^{\frac{1}{2}}}; \text{ hence } p - q\frac{dr}{dx} = 0 \text{ becomes}$$

$$p - q\frac{e^2 x}{(e^2 x^2 - b^2)^{\frac{1}{2}}} = 0;$$

$$\therefore p^2 (e^2 x^2 - b^2) = q^2 e^4 x^2;$$

$$\therefore x^2 = \frac{p^2 b^2}{e^2 (p^2 - q^2 e^2)};$$

$$\therefore x = \frac{pb}{e (p^2 - q^2 e^2)^{\frac{1}{2}}}; \text{ and hence we may find } y, r.$$

Cor. 1. If $q^2 e^2 > p^2$, or $qe > p$, the equilibrium is impossible.

Cor. 2. If $qe = p$, $x = \infty$: the body would in this case be supported upon the asymptote.

16. Prob. II. *It is required to find a curve such that a given weight* = q *hanging over the pully may balance another given weight* = p *at every point of it.*

We must have at every point

$$p - q\frac{dr}{dx} = 0;$$

hence, integrating with respect to x,

$$px - qr + c = 0;$$

$$\therefore px + c = qr = q(x^2 + y^2)^{\frac{1}{2}};$$

$$\therefore p^2 x^2 + 2pcx + c^2 = q^2 x^2 + q^2 y^2;$$

$$\therefore y^2 = \frac{p^2 - q^2}{q^2} x^2 + \frac{2pc}{q^2} x + \frac{c^2}{q^2}.$$

Let $x + \dfrac{pc}{p^2 - q^2} = t$;

$$\therefore (p^2 - q^2) x^2 + 2pcx + \frac{p^2 c^2}{p^2 - q^2} = (p^2 - q^2) t^2,$$

$$y^2 = \frac{1}{q^2}\left\{(p^2 - q^2)\,t^2 - \frac{p^2c^2}{p^2 - q^2} + c^2\right\};$$

$$= \frac{p^2 - q^2}{q^2}\left(t^2 - \frac{q^2c^2}{(p^2 - q^2)^2}\right).$$

But if t, y, be the abscissa and ordinate of a hyperbola in which the semi-axes are a, b,

$y^2 = \dfrac{b^2}{a^2}(t^2 - a^2)$; which agrees with our equation, if

$$\frac{p^2 - q^2}{q^2} = \frac{b^2}{a^2}, \text{ and } \frac{q^2c^2}{(p^2 - q^2)^2} = a^2:$$

$$\text{hence } \frac{c^2}{p^2 - q^2} = b^2.$$

Hence the curve required is a hyperbola in which $KM = x$, and $CM = t$; fig. 99; and in which the semi-axes are

$$a = \frac{qc}{(p^2 - q^2)}, \text{ and } b = \frac{c}{(p^2 - q^2)^{\frac{1}{2}}};$$

$$CK \text{ is } = \frac{pc}{p^2 - q^2} = \frac{pa}{q}.$$

$$(a^2 + b^2)^{\frac{1}{2}} = \left(\frac{q^2c^2}{(p^2 - q^2)^2} + \frac{c^2}{(p^2 - q^2)}\right)^{\frac{1}{2}} = \frac{pc}{p^2 - q^2} = CK;$$

$\therefore$ K is the focus.

If we call AK, k, we have

$$k = CK - CA = \frac{pc}{p^2 - q^2} - \frac{qc}{p^2 - q^2} = \frac{c}{p + q};$$

$$\therefore\ c = (p + q)\,k,$$

and putting this value for c, the semi-axes become

$$a = \frac{q}{p - q}k, \text{ and } b = \left(\frac{p + q}{p - q}\right)^{\frac{1}{2}} k.$$

17. Prop. *Two given weights P, P', connected by a string of given length* ($= \text{b}$) *passing over a given pully K, fig.* 100, *are supported on two curves, which are in the same vertical plane as the pully. Having given one curve, to find the other so that the weights may balance in every position.*

Let the weights be p, p', and the tension of the string q. And let x, x', r, r', be the values of the abscissæ KM, KM', and of KP, KP'. Then since q must be equal to a weight which, hanging freely, would support either P or P', we have by the last Problem,

$$p - q\frac{dr}{dx} = 0, \text{ and } p' - q\frac{dr'}{dx'} = 0.$$

The second equation multiplied by $\frac{dx'}{dx}$ gives

$$p'\frac{dx'}{dx} - q\frac{dr'}{dx} = 0;$$

$$\text{and since } r + r' = b, \quad \frac{dr}{dx} + \frac{dr'}{dx} = 0;$$

whence, if we add the first and the third equations, we have

$$p + p'\frac{dx'}{dx} = 0; \text{ and integrating in } x;$$

$\therefore\ px + p'x' = c$. This equation, along with the one $r + r' = b$, enables us to find x' and r' in terms of x and r: and as we know the nature of the curve $A'P'$, we have the relation between r' and x', which we may represent thus, $r' = f(x')$, $f(x')$ representing a known function of x'; and by substituting the values of x' and r' we have a relation between x and r, which determines the curve required.

18. Prob. III. *As an example, suppose the given curve $A'P'$, fig.* 101, *to be a circle, and CK a vertical line through its center:* and let $KC = k$, $A'C$, the radius of the circle, $= a$; then,

$$KP'^2 = KC^2 + CP'^2 - 2KC \,.\, CM', \text{ or}$$

$$r'^2 = k^2 + a^2 - 2k(k - x') = a^2 - k^2 + 2kx',$$

$$\text{or since } r' = b - r, \text{ and } x' = \frac{c - px}{p'}, \text{ by last Article};$$

$$\therefore\ (b - r)^2 = a^2 - k^2 + \frac{2k}{p'}(c - px),$$

$$\text{or } b - (x^2 + y^2)^{\frac{1}{2}} = \left\{a^2 - k^2 + \frac{2k}{p'}(c - px)\right\}^{\frac{1}{2}}.$$

COR. This is an equation to an epicycloid, as might be shewn. We shall, however, instead of this, shew geometrically that an epicycloid will satisfy the conditions. An epicycloid is the figure described by a point in one circle which *rolls* upon the circumference of another circle, which is fixed.

Let CP', fig. 102, be the radius of the given circle, and K the pully in the vertical line CK. In this line produced take a point O, so that $CK : KO ::$ weight P' : weight P; and in the same line take Oq equal to the length of the string $P'KP$. Take qs equal to qO, and qp equal to qK; and describe a circle qr with center s and radius sq. Let this circle, carrying along with it the point p in the radius sq, produced if necessary, roll along the circle described with center O and radius Oq: the point p will describe a curve pKP, which will possess the property required.

For let qs come into the position QS, so that the describing point p may come to P: if T be the point where the circles are in contact, the circle SQ may, for an instant, be supposed to revolve about the point T, so that the curve will be perpendicular to TP; hence the re-action of the curve will be in the direction PT.

Let SP produced meet CK in y, and let KP be joined: and since, by the description of the curve, the arc TQ is equal to the arc Tq, the angle TSQ is equal to the angle TOq, and therefore yO equal to yS. Also SQ equals sq or Oq, and QP equals qp or qK; hence SP equals OK, and therefore yP equals yK. Hence KP is parallel to OS; and hence if PV be parallel to KO, PV will equal KO; also OV will equal KP.

We made $CK : KO :: P' : P$; hence if KC represent the weight P', OK or VP will represent the weight P. Now the weight P' is kept at rest by three forces, gravity, re-action, and tension, in the directions KC, CP', $P'K$; hence, on this supposition $P'K$ represents the tension of KP'. And the weight P is kept at rest by three forces, gravity, re-action, and tension, in the directions VP, PT, TV; hence TV represents the tension of KP.

Now OT equals KP and KP', and OV equals KP; therefore TV equals KP; and hence the tensions of KP and of KP' are equal, and the bodies will balance each other.*

---

* If instead of supposing a weight P' to rest on the circumference of a circle, we suppose CP' a heavy mass, (as a draw-bridge,) moveable about a hinge at C; and if it be required to find the curve on which P must rest so as always to balance it, the question will easily be seen to be the same. Under this form the Problem was solved by the Marquis de l'Hopital in the Leipsig Acts for February 1695. The curve, which was at first called *the Curve of Equilibration*, was shewn by John Bernoulli to be such an epicycloid as we have proved it to be. From the construction it appears, that if Oq, the length of the string, be to KC as P to P', the curve is the common epicycloid, in which the describing point is in the circumference of the rolling circle or *rota:* if the former ratio be less, as in fig. 102, the describing point is without the circumference of the *rota;* if greater, the describing point is within the circumference of the *rota* and the curve has a point of contrary flexure, as in fig. 101.

The Problems we have solved in the text suggest the following:

PROB. V. *Two weights connected by a string passing over a fixed pully rest on the same curve; to find the nature of the curve that they may in all positions balance.*

By Art. 17, we must have

$$px + p'x' = c, \quad r + r' = b;$$

also r and r' must be the same function of x and x'; that is, if f represent this function,

$$r = f(x), \quad r' = f(x').$$

$$\text{Let } x = t + \frac{c}{p+p'}, \quad x' = t' + \frac{c}{p+p'}, \quad r = u + \frac{b}{2}, \quad r' = u' + \frac{b}{2};$$

whence our equations become

$$pt + p't' = 0, \quad u + u' = 0; \quad \therefore \; t' = -\frac{pt}{p'}; \quad u' = -u.$$

Also u and u' will be the same function of t and t';

$$\therefore \; \phi(t) = -\phi(t') = -\phi\left(-\frac{pt}{p'}\right);$$

from which the form of the function ϕ must be determined, whence the form of f and the nature of the curve will be known.

It does not appear that there exists a solution to this equation, when p and p' are unequal. If $p' = p$, it becomes

$$\phi(t) = -\phi(-t);$$

that is, ϕ must be such a function that it only changes its sign by putting $-t$ for t. This condition will manifestly be satisfied by the functions,

$\phi(t) = mt$,

$\phi(t) =$ any rational function composed of odd powers of t,

$\phi(t) = m \,.\, \sin.\, nt$, &c.

19. PROB. IV. *Fig.* 103. *A body P, which hangs by a string CP, without weight, and consequently must be somewhere in the circumference whose center is C, is sustained at the point P by a repulsion acting from the lowest point A. The repulsive force is directly proportional to the intensity of the repulsive power in A and inversely proportional to the square of the distance. Knowing the repulsive power, to find the position; and conversely, from the position, to find the intensity of the repulsive power.*

An instrument of this kind is used to measure the intensity of the electrical repulsions which exist between two bodies A and P, in the same state of electricity; and it is then called the *Electrometer*.

Let $CA = CP = a$, $AP = r$; NP, perpendicular on CA, $= y$; $AN = x$. And let f represent the intensity of the repulsive power of A: then the force which it exerts at the distance r will be proportional to $\frac{f}{r^2}$; and if f be equal to the force at a distance $= 1$, $\frac{f}{r^2}$ will be *equal* to the force

---

If we make $\phi(t) = mt$, we have $u = mt$;

$$\text{or } r - \frac{b}{2} = mx - \frac{mc}{2p};$$

$$\text{or } (x^2 + y^2)^{\frac{1}{2}} = mx - \frac{b}{2} - \frac{mc}{2p}$$

which will give a hyperbola as in Prob. V. In fact, it is manifest that since equal weights in two positions P and P', would each support a weight Q, they will support each other; and this in every situation.

If we make $\phi(t) = m \,.\, \text{sin.}\, nt$, we have $u = m \,.\, \text{sin.}\, nt$;

$$\text{or } r - \frac{b}{2} = m \,.\, \text{sin.}\, n\left(x - \frac{c}{2p}\right):$$

and if we now suppose, that when $x' = 0$, x is $= 2h$, we have $c = 2ph$; and hence

$$r = \frac{b}{2} + m \text{ sin. } n(x - h).$$

When $x = h$, $r = \frac{b}{2}$; $\therefore r' = \frac{b}{2}$, and $x' = h$. Hence if with radius $KB = \frac{b}{2}$, fig. 104, we describe a circle, and take $KH = h$ in the vertical line, and draw DH horizontal, D, D, will be corresponding positions of the weights. When one is at E the other will be at F; and in other positions P, P' they will rest on such a curve as is represented in the figure.

at P, in the direction AP. Resolve this force in the directions AN, NP, or PX, PY, and the forces will be

$$\frac{f}{r^2}\cdot\frac{x}{r}, \text{ and } \frac{f}{r^2}\cdot\frac{y}{r}.$$

Hence, considering also the action of gravity $= p$, we have, (see Art. 13.)

$$X = \frac{fx}{r^3} - p, \quad Y = \frac{fy}{r^3}.$$

But, in the circle, $y^2 = 2ax - x^2$; $\therefore\ y\frac{dy}{dx} = a - x$.

Hence the formula $X + Y\frac{dy}{dx} = 0$, or $Xy + Yy\frac{dy}{dx} = 0$,

becomes $Xy + Y(a - x) = 0$.

And putting for X and Y their values;

$$\frac{fxy}{r^3} - py + \frac{fay}{r^3} - \frac{fyx}{r^3} = 0,$$

or $\frac{fa}{r^3} = p$; $\therefore\ r = \left(\frac{fa}{p}\right)^{\frac{1}{3}}$; whence the position is known.

And $\frac{f}{p} = \frac{r^3}{a}$, whence the ratio of the force f to the weight p is known.

Cor. If another force of the same kind, $(=f')$ balance the same body P, at a distance r' from A, we have also

$$\frac{f'}{p} = \frac{r'^3}{a}; \quad \therefore \frac{f'}{f} = \frac{r'^3}{r^3};$$

or the forces are as the cubes of the distances from A at which the body is supported.

The Principle of Virtual Velocities.

20. In the Elementary Treatise, Articles 45 to 57, the following proposition was demonstrated, upon the suppositions which were there adopted:

In any of the simple machines, the power is to the weight, as the weight's velocity in the direction of its action is to the power's velocity in the direction of its action.

This proposition was proved by shewing its truth in the case of each of the mechanical powers separately. In some of the cases, the power was supposed to act in a direction parallel to that in which the weight acts, both forces being supposed to be the result of gravity; as in the case of the toothed-wheels, Art. 48, and the pullies, Art. 49, &c. But this supposition does not affect the result; and if we suppose the attractive force which produces the power in these cases to act in a direction different from the direction of gravity which acts on the weight, the proposition will still appear to be true by the same reasoning. The string by which the power acts, will be drawn in the direction of the accelerating force from which the power results, and both the power itself and the space described by the power, corresponding to a given space described by the weight, will be the same as they were in the demonstration to which we refer; and therefore the validity of the demonstration will not be disturbed by such a change of supposition.

21. But any number of points connected in any manner may be considered as a machine, and the proposition thus proved for simple machines is capable of being extended to this more general case. Each of the points may be so constrained in its motions by strings, rods, and surfaces, that it is not at liberty to move otherwise than in a certain curve or curve surface: and any two of the points may be connected by strings or rods so as to affect each other's motions. The construction of the machine may be such, that the strings may have to move through fixed rings or over fixed pullies; and in this way the motion of one point will constrain the motion of one or more of the others; and there will be mutual forces exerted among the points. But these mutual forces must be such as destroy each other: for instance, if two points be connected by a string, the pressures which each point exerts upon the other, by means of the string, must be equal and opposite: and the same will be true of the forces exerted by means of rigid rods. Now there may be obtained a relation among the external forces which act on the machine, independent of the internal forces or mutual pressures; and the expression of this relation forms the ex-

tension of the proposition in Art. 45. of the Elementar
Treatise of which we have spoken.

This proposition thus extended is called *the Princip*
of Virtual Velocities; it may be expressed as follows.

22. Let P, Q, R, &c. be any forces which, acting
certain points of any machine whatever, balance each othe
and let α, β, γ, &c. be respectively the spaces through whi
the points at which these forces act can move simultaneousl
the relation of these spaces being determined by the co
struction of the machine, and the spaces being consider
as negative, when the points move in directions opposite
the forces. This being supposed, and the spaces α, β, γ, &
having the ratio to which they tend when they are indefinit
diminished, we shall have

$$P\alpha + Q\beta + R\gamma + \&c. = 0.$$

In order to prove this, let it be observed in the first pl
that α, β, γ, &c. the spaces described by the points at
the forces act, are, when they are indefinitely dimin
the velocities of these points. We shall suppose
thus diminished; and shall call them the *virtual*
the points to which they belong.

If we had only *two* forces, P and
chine, we should have, by Art

$$P : Q$$

one of the quantities α,
one point moves in
it, the other poin
force which acts

Sup
and
wh

...*sultant of any number of* ...*id body. Fig.* 105.

...llel forces p_1, p_2, p_3, &c. act ...plane yAx be drawn, perpendicular ...n lines, Ax, and Ay, be drawn ...gles to each other. Let P_1M_1 be ...y_1 be AM_1, M_1P_1, the *co-ordinates* ...p_1 meets the plane yAx. Similarly ...nates of P_2, where p_2 meets the plane; ...ntities for P_3, &c. And let R be the ... and a, β, the co-ordinates of the point ...plane.

...p_1, p_2, produce the same effect as if they ...And if we consider them as weights, they ...h other upon their center of gravity, and ...point a pressure $= p_1 + p_2$. (*Elem. Tr.*) Hence ...n the rigid body is the same as that of a force ...the center of gravity of P_1, P_2, in a direction ...forces. Similarly it will appear that this ...g with p_3, that is, p_1, p_2, p_3, will produce ...$p_1 + p_2 + p_3$, acting at the center of gravity ...And in the same manner it may be shewn ...mber of forces p_1, p_2, p_3, &c. will produce the ...s $p_1 + p_2 + p_3 +$ &c. acting parallel to the forces, ...ter of gravity of P_1, P_2, P_3, &c.

...ce we shall have, by the properties of the center of (*Elem. Tr.*)

$$R = p_1 + p_2 + p_3 + \&c.$$
$$Ra = p_1x_1 + p_2x_2 + p_3x_3 + \&c.$$
$$R\beta = p_1y_1 + p_2y_2 + p_3y_3 + \&c.$$

This formula expresses *the Principle of Virtual Velocities*. It enables us to put into an equation the conditions of equilibrium of any system of forces whatever, anyhow connected.

For this purpose we may consider two possible positions of the system indefinitely near each other; and find general expressions for α, β, γ, &c. introducing into these expressions as many indeterminate quantities as there are arbitrary elements in the variation of position of the system. These expressions are to be substituted for α, β, γ, δ in the above equation. And this equation must be true independently of all the indeterminate quantities, in order that the equilibrium may subsist in general, and that motion may not take place in any direction. We must therefore take the sum of the terms which involve each of the indeterminate quantities, and make each sum separately equal to nothing. In this way we shall have as many distinct equations as there are indeterminate quantities. This will also be the same number as that of the unknown quantities in the position of the system; we shall therefore have thus as many equations as are requisite to determine the position of the system.

But when the point at which Q acts moves through a space perpendicular to the direction of Q, we may suppose the motion of this point to be constrained, and $\beta = 0$; whence

$$P\alpha + R\gamma = 0,$$

by what has been already said. Therefore

$$M + M' = 0,$$

or the forces which constrain the points at which Q', Q'' act, in the systems P, Q', and Q'', R, destroy each other when the two systems are combined; and therefore the point at which $Q' + Q''$ acts, is then acted upon by that force, that is, by the force Q, and by no other; whence the demonstration above given is valid.

CHAP. III.

THE CONDITIONS OF EQUILIBRIUM OF A RIGID BODY.

23. Prop. *To find the resultant of any number of parallel forces acting on a rigid body.* *Fig.* 105.

Let any number of parallel forces p_1, p_2, p_3, &c. act upon a rigid body. Let a plane yAx be drawn, perpendicular to these forces; and let two lines, Ax, and Ay, be drawn in this plane at right angles to each other. Let P_1M_1 be parallel to Ay, and let x_1, y_1 be AM_1, M_1P_1, the *co-ordinates* of the point P_1, where p_1 meets the plane yAx. Similarly let x_2, y_2, be the co-ordinates of P_2, where p_2 meets the plane; x_3, y_3, the same quantities for P_3, &c. And let R be the resultant of the forces, and a, β, the co-ordinates of the point where it meets the plane.

The two forces p_1, p_2, produce the same effect as if they acted at P_1, P_2. And if we consider them as weights, they will balance each other upon their center of gravity, and produce at that point a pressure $= p_1 + p_2$. (*Elem. Tr.*) Hence their effect upon the rigid body is the same as that of a force $p_1 + p_2$ acting at the center of gravity of P_1, P_2, in a direction parallel to these forces. Similarly it will appear that this force $p_1 + p_2$, along with p_3, that is, p_1, p_2, p_3, will produce the same effect as $p_1 + p_2 + p_3$, acting at the center of gravity of P_1, P_2, P_3. And in the same manner it may be shewn that any number of forces p_1, p_2, p_3, &c. will produce the same effect as $p_1 + p_2 + p_3 +$ &c. acting parallel to the forces, at the center of gravity of P_1, P_2, P_3, &c.

Hence we shall have, by the properties of the center of gravity (*Elem. Tr.*)

$$R = p_1 + p_2 + p_3 + \text{\&c.}$$
$$Ra = p_1x_1 + p_2x_2 + p_3x_3 + \text{\&c.}$$
$$R\beta = p_1y_1 + p_2y_2 + p_3y_3 + \text{\&c.}$$

Whence R is known, and a, β, which determine the position of the resultant.

COR. 1. If any of the forces act in the opposite direction they must be considered as negative.

Hence it appears that we may have $p_1 + p_2 + p_3 +$ &c. $= 0$. In this case, if $p_1x_1 + p_2x_2 +$ &c. be finite, a will be infinite. And similarly for β.

COR. 2. For example, let two forces, each $= p$, act in opposite directions at points in the line Ax, distant from each other by a distance a. Hence we shall have

$$R = 0; \quad Ra = p(x_1 + a) - px_1 = pa;$$

$$\beta = 0; \quad a = \frac{pa}{R} = \frac{pa}{0}.$$

Therefore a is infinite, and the forces are equivalent to a force $= 0$, acting at any infinite distance.

In this case no single force could produce the effect of the two. Their tendency is to turn the system round in the plane in which they are, without producing any motion except a rotatory one.

COR. 3. Hence it is not true that parallel forces can in all cases be reduced to a single finite force. If $p_1+p_2+p_3+$&c.$=0$, they can not.

In this case the forces can be reduced to two, equal to each other, and acting at two different points in opposite directions. For since $p_1 + p_2 +$ &c. $= 0$, these forces may be divided into two groups, of which one is equal to the other with a negative sign. And hence if we take the resultants of these groups separately, we shall obtain two equal forces in opposite directions.

COR. 4. Let one of these groups consist of p_1, p_2, &c. and the other of p', p'', &c. Then

$$p_1 + p_2 + \text{\&c.} + p' + p'' + \text{\&c.} = 0,$$

$$p_1 + p_2 + \text{\&c.} = -p' - p'' - \text{\&c.}$$

And if R be the resultant of p_1, p_2, &c., $-R$ will also be the resultant of p', p'', &c. Let α, β, be the co-ordinates of the point where the first resultant meets the plane, α', β', the corresponding co-ordinates for the second resultant. Then

$$R\alpha = p_1x_1 + p_2x_2 + \&c.;\quad R\beta = p_1y_1 + p_2y_2 + \&c.$$
$$-R\alpha' = p'x' + p''x'' + \&c.; -R\beta' = p'y' + p''y'' + \&c.$$
$$\therefore R(\alpha - \alpha') = p_1x_1 + p_2x_2 + \&c. + p'x' + p''x'' + \&c.$$
$$R(\beta - \beta') = p_1y_1 + p_2y_2 + \&c. + p'y' + p''y'' + \&c.$$

And the quantities on the right hand are the same whatever be R, that is, however the groups are selected. If l be the line which joins the two points of application, λ the angle which it makes with Ax,

$$Rl = R\sqrt{\{(\alpha - \alpha')^2 + (\beta - \beta')^2\}}\ ;\ \tan.\lambda = \frac{\beta - \beta'}{\alpha - \alpha'};$$

and these quantities are the same whatever R be.

Hence the position of the line l, and the moment of the pair of forces to turn the system in the plane in which are R and l, are found to be the same, however the two groups are selected.

24. Prop. *To find the conditions of equilibrium of parallel forces acting upon a rigid body.*

In order that the equilibrium may subsist, one of the forces, as p_1, must be equal and opposite to the resultant of all the others. Hence $-p_1$ must be the resultant of the forces p_2, p_3, &c. And therefore by last Article, x_1, y_1, x_2, y_2, &c. being the co-ordinates, as before,

$$-p_1 = p_2 + p_3 + \&c.$$
$$-p_1x_1 = p_2x_2 + p_3x_3 + \&c.$$
$$-p_1y_1 = p_2y_2 + p_3y_3 + \&c.;$$
$$\therefore p_1 + p_2 + p_3 + \&c. = 0\ldots\ldots\ldots\ldots\ldots(1);$$
$$\left.\begin{array}{l} p_1x_1 + p_2x_2 + p_3x_3 + \&c. = 0 \\ p_1y_1 + p_2y_2 + p_3y_3 + \&c. = 0 \end{array}\right\}\ldots\ldots\ldots(2).$$

Cor. 1. If the rigid body have one point fixed, let this point be the origin of co-ordinates. And it is manifest that the equilibrium will subsist if the resultant pass through this point; for it will be counteracted by the resistance of the fixed point. Hence in last Article $a = 0$, $\beta = 0$. Therefore, by that Article,

$$p_1x_1 + p_2x_2 + p_3x_3 + \&c. = 0;$$

$$p_1y_1 + p_2y_2 + p_3y_3 + \&c. = 0.$$

25. Prop. *To find the resultant of any number of forces acting in the same plane upon a rigid body. Fig.* 106.

Let p_1, p_2, p_3, &c. be the forces acting in the plane yAx, at the points P_1, P_2, P_3, &c.: let Ax, Ay be at right angles in this plane; and let x_1, y_1; x_2, y_2; x_3, y_3, &c. be the co-ordinates of these points parallel to Ax, Ay. Let P_1D_1, P_1E_1 be lines parallel to Ax, Ay, and let a_1, be the angle which P_1p_1 makes with P_1D_1. If the force p_1 be resolved in these directions, the components will be $p_1 \cos. a_1$, $p_1 \sin a_1$. In the same manner $p_2 \cos. a_2$, $p_2 \sin. a_2$ will be the components of p_2; and similarly for the others. Hence the forces are thus resolved into two sets of parallel forces, acting at the points P_1, P_2, P_3, &c. and parallel to Ax and to Ay respectively. Let X be the resultant of the first set; Y, of the second. Also let X meet Ay in K, and let $AK = t$; and let Y meet Ax in H, and let $AH = s$. Then we may suppose X to act at K, and Y at H. Therefore, by Art. 23,

$$X = p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c.$$

$$Y = p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 + \&c.$$

$$Ys = p_1x_1 \sin. a_1 + p_2x_2 \sin. a_2 + p_3x_3 \sin. a_3 + \&c.$$

$$Xt = p_1y_1 \cos. a_1 + p_2y_2 \cos. a_2 + p_3y_3 \cos. a_3 + \&c.$$

Hence we know X, Y, s, t. And knowing AK, AH, if we draw KG, HG parallel to Ax, Ay, the forces X, Y may be supposed to act at G; and will then produce a resultant R, which is the resultant of the whole system. Also if a be the angle which this resultant makes with Ax, we shall have

$$R = \sqrt{(X^2 + Y^2)}; \ \tan. a = \frac{Y}{X}.$$

Hence we know the magnitude and position of R. The co-ordinates s, t, of its point of application have already been found.

Cor. 1. To find the equation of the straight line in which R acts.

Let the equation to the straight line be $y = Ax + B$. In this case A is the tangent of the angle α; therefore $A = \frac{Y}{X}$,

$$\text{and } y = \frac{Y}{X}x + B.$$

Also, since G is a point in this line, when $x = s$, $y = t$.

Therefore $t = \frac{Y}{X}s + B$; and $B = t - \frac{Y}{X}s$. Hence

$$y = \frac{Y}{X}x + t - \frac{Y}{X}s; \text{ and } Xy - Yx = Xt - Ys,$$

which is the equation to the line.

Cor. 2. Putting for Xt and Ys their values, we have

$$Xy - Yx = p_1(y_1 \cos.\alpha_1 - x_1 \sin.\alpha_1)$$
$$+ p_2(y_2 \cos.\alpha_2 - x_2 \sin.\alpha_2) + \&c.$$

call this quantity L. Then the equation is

$$Xy - Yx = L.$$

Cor. 3. In this case, when *one* of the sets of parallel forces, as that parallel to Ax, is not reducible to a single force, (see Cor. 3, Art. 23,) we shall have $X = 0$, $t = \text{inf.}$ Hence

$$R = Y, \quad \cos.\alpha = 0, \quad s = \frac{p_1 x_1 \sin.\alpha_1 + p_2 x_2 \sin.\alpha_2 + \&c.}{Y}.$$

Hence the resultant will be a single force parallel to Ay, and determined in magnitude and position by the above equations.

COR. 4. But in the case when *both* the sets of parallel forces are incapable of being reduced to single forces, we shall have $X=0$, $Y=0$, $s=\text{inf.}$ $t=\text{inf.}$ Hence $R=0$; and we have a resultant $=0$, acting at an infinite distance.

In this case the forces are equal to two, equal and opposite, but not in the same line, as in Cor. 3, Art. 23.

26. PROP. *To find the conditions of equilibrium of any number of forces acting in the same plane upon a rigid body.*

In order that there may be an equilibrium, p_1 must be equal and opposite to the resultant of p_2, p_3, &c. Hence $-p_1$ must be the resultant of p_2, p_3, &c. And $-p_1 \cos. a_1$, $-p_2 \sin. a_2$ will be the parts of the resultant which are parallel to Ax, Ay. Hence, by the equations of Art. 23,

$$-p_1 \cos. a_1 = p_2 \cos. a_2 + p_3 \cos. a_3 + \&c.$$

$$-p_1 \sin. a_1 = p_2 \sin. a_2 + p_3 \sin. a_3 + \&c.$$

Hence

$$\left.\begin{array}{l} p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c. = 0 \\ p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 + \&c. = 0 \end{array}\right\} \ldots\ldots(1).$$

Also the equation of the line in which $-p_1$ acts must be the same as that in which the resultant of p_2, p_3, &c. acts. And the latter equation is, (Cor. 2, Art. 25,) putting for X, $-p_1 \cos. a_1$, and for Y, $-p_1 \sin. a_1$,

$$-p_1 y \cos. a_1 + p_1 x \sin. a_1 = p_2 (y_2 \cos. a_2 - x_2 \sin. a_2)$$
$$+ p_3 (y_3 \cos. a_3 - x_3 \sin. a_3) + \&c.$$

And this, which is true for every point of p_1's direction, must be true for the point P_1, where $x = x_1$, $y = y_1$. Hence, putting these values for x and y, and transposing,

$$p_1 (y_1 \cos. a_1 - x_1 \sin. a_1) + p_2 (y_2 \cos. a_2 - x_2 \sin. a_2)$$
$$+ p_3 (y_1 \cos. a_3 - x_3 \sin. a_3) + \&c. = 0 \ldots\ldots\ldots(2).$$

This equation (2) and the two found above (1) are the equations of condition for the equilibrium of the forces.

COR. 1. If a point of the rigid body, in the plane in which the forces act, be a fixed point, the equilibrium will subsist if the resultant of the forces pass through this point; for the effect of the force will be balanced by the re-action of the fixed point.

Let A be the fixed point. Then, in order that the resultant may pass through A, we must have $x = 0$, $y = 0$, at the same time;

$\therefore 0 = Xt - Ys,$

or $0 = p_1 (y_1 \cos. \alpha_1 - x_1 \sin. \alpha_1) + p_2 (y_2 \cos. \alpha_2 - x_2 \sin. \alpha_2) + \&c.$

which is in this case the condition of equilibrium.

27. PROP. *Any number of forces being given, acting in any directions upon a rigid body, to reduce them to two sets of forces, one set being in a given plane, and the other perpendicular to it.*

Let Ax, Ay, Az, fig. 107, be three lines at right angles to each other. Let P be a point of the system, at which one of the forces acts, in the direction Pp. Let PD, PE, PF be three lines parallel to Ax, Ay, Az, and let α, β, γ be the angles which Pp makes with PD, PE, PF. Then p being the force, $p \cos. \alpha$, $p \cos. \beta$, $p \cos. \gamma$ will be the components in PD, PE, PF. Let FP meet the plane yAx in O, and let OM and ON be parallel to Ay and Ax.

If we suppose, at the point P, two equal forces in opposite directions to be added to the system, these will counteract each other, and the effect of the forces such as p will be the same as before. Let two forces, g in the direction PF, and g in the direction FP, act at P. Then the forces which act at P may be grouped thus,

$$p \cos. \alpha \text{ and } g; \quad p \cos. \beta \text{ and } -g; \quad p \cos. \gamma.$$

Let $p \cos. \alpha$ in PD, and g in PF have a resultant Ph: hP will be in the plane DPF: let it meet ON in H. Ph acting at P is equivalent to Ph acting at H. And Ph at H may be resolved in two forces, $p \cos. \alpha$ parallel to Ax, and g parallel to Az.

Since Ph is compounded of p cos. α in the direction HO, and g in the direction OP, we shall have, calling the co-ordinates of P, x, y and z,

$$HO : OP (= z) :: p \cos. \alpha : g;$$

$$\therefore HO = \frac{pz \cos. \alpha}{g}. \quad \text{And } NH = NO - HO = x - \frac{pz \cos. \alpha}{g}.$$

Hence p cos. α and g at P, are equivalent to g parallel to Az, and p cos. α parallel to Ax; both acting at a point H, of which the co-ordinates parallel to

$$Ax \text{ and } Ay, \text{ are } x - \frac{pz \cos. \alpha}{g}, \text{ and } y.$$

In the same manner p cos. β and $-g$ are equivalent to a force Pk in the plane EPF, and this produces the same effect as if it acted at K. And at K it may be resolved into

$$p \cos. \beta \text{ and } -g.$$

Also as before,

$$OK = \frac{pz \cos. \beta}{g}; \text{ and } MK = y + \frac{pz \cos. \beta}{g}.$$

Hence p cos. β and $-g$ are equivalent to p cos. β parallel to Ay and $-g$ parallel to Az; both acting at a point K of which the co-ordinates parallel to

$$Ax \text{ and } Ay, \text{ are } x \text{ and } y + \frac{pz \cos. \beta}{g}.$$

p cos. γ is parallel to Az, and produces the same effect as if it acted at O, of which the co-ordinates are x, y.

Hence if p_1 is a force acting at a point of which the co-ordinates are x_1, y_1, z_1, making with the three co-ordinate angles α_1, β_1, γ_1; and if p_2, p_3, &c. be other forces; x_2, y_2, z_2; x_3, y_3, z_3, &c. the corresponding co-ordinates; α_2, β_2, γ_2; α_3, β_3, γ_3, &c. the corresponding angles; the forces p_1, p_2, p_3, &c. will be equivalent to the following forces in the plane yAx;

$p_1 \cos. \alpha_1$, par. to Ax, with co-ordinates $x_1 - \dfrac{p_1 z_1 \cos. \alpha_1}{g_1}$ and y_1;

$p_1 \cos. \beta_1$, par. to Ay, with co-ordinates x_1 and $y_1 + \dfrac{p_1 z_1 \cos. \beta_1}{g_1}$;

and to the following forces parallel to Az,

$p_1 \cos. \gamma_1$ with co-ordinates x_2, y_1;

g_1 with co-ordinates $x_1 - \dfrac{p_1 z_1 \cos. \alpha_1}{g_1}$ and y_1;

$-g_1$ with co-ordinates x_1 and $y_1 + \dfrac{p_1 z_1 \cos. \beta_1}{g_1}$.

And to similar forces with the exponents 2, 3, &c. instead of 1.

28. Prop. *To find the conditions of equilibrium of any number of forces p_1, p_2, p_3, &c. acting in any directions upon a rigid body.*

The forces being resolved as in the last Article, the equilibrium will subsist if the forces in the plane yAx, and the forces parallel to Az be in equilibrium separately. Hence we shall have by Art. 26, these three equations for the equilibrium of the forces in the plane yAx:

$$\left.\begin{array}{l} p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \text{\&c.} = 0 \\ p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 + \text{\&c.} = 0 \end{array}\right\} \ldots\ldots (1);$$

$$p_1 (y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + p_2 (y_2 \cos. \alpha_2 - x_2 \cos. \beta_1)$$
$$+ p_3 (y_3 \cos. \alpha_3 - x_3 \cos. \beta_3) + \text{\&c.} = 0 \ldots\ldots\ldots (2).$$

Also by Art. 24, we have, for the equilibrium of the forces parallel to Az,

$$p_1 \cos. \gamma_1 + \text{\&c.} = 0;$$

$$p_1 x_1 \cos. \gamma_1 + g_1 \left(x_1 - \frac{p_1 z_1 \cos. \alpha_1}{g_1}\right) - g_1 x_1 + \text{\&c.} = 0;$$

$$p_1 y_1 \cos. \gamma_1 + g_1 y_1 - g_1 \left(y_1 - \frac{p_1 z_1 \cos. \beta_1}{g_1}\right) + \text{\&c.} = 0;$$

with other similar terms corresponding to p_2, g_2, p_3, g_3, &c. And these three latter equations become

$$p_1 \cos.\gamma_1 + p_2 \cos.\gamma_2 + p_3 \cos.\gamma_3 + \&c. = 0 \ldots\ldots\ldots (1);$$

$$\left.\begin{array}{l} p_1 (x_1 \cos.\gamma_1 - z_1 \cos.\alpha_1) + p_2 (x_2 \cos.\gamma_2 - z_2 \cos.\alpha_2) \\ \qquad + p_3 (x_3 \cos.\gamma_3 - z_3 \cos.\alpha_3) + \&c. = 0, \\ p_1 (y_1 \cos.\gamma_1 - z_1 \cos.\beta_1) + p_2 (y_2 \cos.\gamma_2 - z_2 \cos.\beta_2) \\ \qquad + p_3 (y_3 \cos.\gamma_3 - z_3 \cos.\beta_3) + \&c. = 0. \end{array}\right\} \ldots (2).$$

which these three equations, with the former three, are the conditions of equilibrium.

COR. 1. It has been proved that these equations are *sufficient;* that is, that if they are satisfied the equilibrium subsists. They are also *necessary;* for except both sets are satisfied the equilibrium does not subsist.

If possible, let the equilibrium subsist when the forces parallel to z are not separately in equilibrium. The equilibrium will still subsist if we suppose any line in the plane yAx to be fixed. But in that case, all the forces in the plane yAx will be counteracted by the resistance of this line. And the forces parallel to Az will turn the system about this line in some of its positions. Hence the equilibrium will not subsist.

And since the forces parallel to Az are in equilibrium separately, the other forces must also be in equilibrium separately.

COR. 2. Let the rigid body be moveable about a fixed point. Let this point be made the origin of co-ordinates A. Then the forces may be resolved as in Art. 27. and the equilibrium will subsist if the forces in the plane yAx have a resultant which passes through the point A, and if the forces parallel to Az have also a resultant which passes through A. Hence by Art. 25, Cor. 1, and by Art. 24, Cor. 1, we have

$$p_1 (y_1 \cos.\alpha_1 - x_1 \cos.\beta_1) + p_2 (y_2 \cos.\alpha_2 - x_2 \cos.\beta_2) + \&c. = 0;$$

$$p_1 (x_1 \cos.\gamma_1 - z_1 \cos.\alpha_1) + p_2 (x_2 \cos.\gamma_2 - z_2 \cos.\alpha_2) + \&c. = 0;$$

$$p_1 (y_1 \cos.\gamma_1 - z_1 \cos.\beta_1) + p_2 (y_2 \cos.\gamma_2 - z_2 \cos.\beta_2) + \&c. = 0.$$

It appears from this, that the forces are to be such as to keep each other in equilibrium about three axes at right angles to each other passing through the fixed point.

29. PROP. *To find the condition which is requisite in order that a system of forces acting anyhow in space may have a single resultant.*

Retaining the notation of Art. 27, we may reduce the forces to the two sets mentioned in that Article. The resultants of these sets may be found by Articles 23 and 25; and if these resultants intersect each other, they may be compounded into a single force which will be the resultant of the whole. If the two resultants do not intersect each other, this will be impossible.

$$\text{Let } p_1 \cos.\alpha_1 + p_2 \cos.\alpha_2 + \&c. = X;$$
$$p_1 \cos.\beta_1 + p_2 \cos.\beta_2 + \&c. = Y;$$
$$p_1 \cos.\gamma_1 + p_2 \cos.\gamma_2 + \&c. = Z.$$
$$p_1 (y_1 \cos.\alpha_1 - x_1 \cos.\beta_1) + p_2 (y_2 \cos.\alpha_2 - x_2 \cos.\beta_2) + \&c. = L;$$
$$p_1 (x_1 \cos.\gamma_1 - z_1 \cos.\alpha_1) + p_2 (x_2 \cos.\gamma_2 - z_2 \cos.\alpha_2) + \&c. = M;$$
$$p_1 (z_1 \cos.\beta_1 - y_1 \cos.\gamma_1) + p_2 (z_2 \cos.\beta_2 - y_2 \cos.\gamma_2 + \&c. = N^*.$$

Then we shall have for the equation of the line in which the force acts, which is the resultant of those in the plane yAx,

$$Xy - Yx = L; \text{ (Cor. 2, Art. 25.)}$$

and for the force which is the resultant of those parallel to Az we shall have, by Art. 23, (as in Art. 28.)

$$Zx = M; \ Zy = -N.$$

In order that these two forces may intersect, the point in which the latter meets the plane yAx must be in the line of the direction of the former. Hence the equation $Xy - Yx = L$

---

* The quantities L, M, N are the *moments* of the forces p_1, p_2, &c. *projected* on the planes xy, xz, yz respectively. These projected moments give rise to some remarkable propositions. See Poisson, *Traité de Mec.* Liv. III. Chap. II.

must be satisfied by the values of x and y in $Zx = M$, $Zy = -N$; and substituting, we find

$$LZ + MY + NX = 0 \ldots\ldots (a),$$

which is the equation of condition required.

30. PROP. *In the case where it is possible, to find the resultant of any number of forces acting anyhow in space.*

The force in the plane yAx will be composed of X and Y, and the force at the same point parallel to Az will be Z. Hence, if R be the resultant, and a, b, c, the angles which it makes with lines parallel to Ax, Ay, Az, we shall have, as in Art. 11,

$$R = \sqrt{(X^2 + Y^2 + Z^2)};$$

$$\cos. a = \frac{X}{R}; \ \cos. b = \frac{Y}{R}; \ \cos. c = \frac{Z}{R}.$$

And the point where the resultant cuts the plane yAx is known by the equations $Zx = M$, $Zy = -N$.

COR. It may be easily shewn that the equations to the line in which the resultant acts are

$$Xy - Yx = L, \ Zx - Xz = M, \ Yz - Zy = N \ldots\ldots (b),$$

of which two only are necessary, the third being included in them in consequence of the equation of condition (a) of last Article.

31. PROP. *In the case where a number of forces are not reducible to one force, they are always reducible to two.*

Without altering the conditions of the system, we may suppose, in addition to the forces of the system, two new forces S, $-S$, acting at the origin A, and making angles a, b, c, with the axes. And these forces and their angles may be so taken that the force S, along with p_1, p_2, &c., shall satisfy the equation (a), and have a single resultant. Thus the forces are reduced to this resultant, and to the force $-S$ acting at the point A.

COR. In this case the two forces to which the system is reduced are not determined in magnitude and direction.

32. The following example may serve to illustrate the preceding Articles.

ABCDEFG, fig. 108, is a rectangular parallelepiped acted on by forces, which have their directions in the edges *BE*, *CF*, *DG* of the parallelepiped, taken so that none of them pass through *A*, and no two of them are in the same plane: to shew when there is a single resultant, and to find it.

Let *AD*, *AB*, *AC* be in the directions of Ax, Ay, Az; let $AD = a$, $AB = b$, $AC = c$: and let the forces be p_1, p_2, p_3. Then we shall have

$$p_1 \cos. \alpha_1 = p_1, \quad p_1 \cos. \beta_1 = 0_1, \quad p_1 \cos. \gamma_1 = 0;$$

$$p_2 \cos. \alpha_2 = 0, \quad p_2 \cos. \beta_2 = p_2, \quad p_2 \cos. \gamma_2 = 0;$$

$$p_3 \cos. \alpha_3 = 0, \quad p_3 \cos. \beta_3 = 0, \quad p_3 \cos. \gamma_3 = p_3.$$

$$x_1 = 0, \quad y_1 = b, \quad z_1 = 0;$$

$$x_2 = 0, \quad y_2 = 0, \quad z_2 = c;$$

$$x_3 = a, \quad y_3 = 0, \quad z_3 = 0;$$

Hence we have $X = p_1, \quad Y = p_2, \quad Z = p_3$

$$L = p_1 b; M = p_3 a, N = p_2 c.$$

And the equation of condition (a) becomes

$$p_1 p_3 b + p_2 p_3 a + p_1 p_2 c = 0,$$

$$\text{or } \frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} = 0.$$

If this equation be not satisfied, the forces are not reducible to a single force.

Let p_1, p_2, be as the edges *BE*, *CF*. Then

$\dfrac{b}{p_2} = \dfrac{a}{p_1}$; $\therefore \dfrac{c}{p_3} = -\dfrac{2a}{p_1}$ when the reduction is possible;

$$\therefore L = p_1 b; \; M = p_3 a = -\tfrac{1}{2} p_1 c; \; N = p_2 c = p_1 \frac{bc}{a}.$$

Hence, by Cor. to Art. 30. the equations to the line of direction of the force will be

$$\left.\begin{aligned} p_1 y - p_2 x = p_1 b, \text{ or } y - \frac{b}{a} x = b \\ p_3 x - p_1 z = -\tfrac{1}{2} p_1 c, \text{ or } \frac{cx}{a} + 2z = c \end{aligned}\right\}.$$

These two equations determine the position of the force; and its magnitude is known, being

$$= \sqrt{(p_1^2 + p_2^2 + p_3^2)} = p_1 \frac{\sqrt{(a^2 + b^2 + \frac{1}{4} c^2)}}{a}.$$

If we produce DE to H, making $EH = ED$, BH will be the line to which the first equation belongs. And when

$$x = 0, \; z = \tfrac{1}{2} c.$$

Hence, if we bisect BF in K, KH is the direction in which the resultant of the three forces acts.

CHAP. IV.

THE APPLICATION OF THE INTEGRAL CALCULUS TO FINDING THE CENTER OF GRAVITY.

33. Prop. *To find the center of gravity of any curvilinear body.*

Let $P'PBQQ'$ (fig. 109.) be any body: Ax the axis of x: and let the body be cut by planes PQ, $P'Q'$, perpendicular to Ax.

Let G, G', K, be the centers of gravity of the portions of the body PBQ, $P'BQ'$, and $PQQ'P'$; and let GH, $G'H'$, KL, be perpendiculars upon a plane Ay parallel to PM.

Let the mass $PBQ = m$, $P'BQ' = m'$; therefore we have $PQQ'P' = m' - m$.

Also let $GH = h$, $G'H' = h'$, $KL = k$.

Now we may suppose the masses PBQ, $PQQ'P'$, to be collected at their respective centers of gravity G, K: and since G' is the center of gravity of the whole mass, we have (*Elem. Tr.*)

$$G'H' = \frac{PBQ \,.\, GH + PQQ'P' \,.\, KL}{PBQ + PQQ'P'};$$

$$\text{or } h' = \frac{mh + (m' - m)\,k}{m'}.$$

$$\text{Hence, } k = \frac{m'h' - mh}{m' - m}.$$

Now, if we suppose the plane $P'Q'$ to come indefinitely near to PQ, so that $PQQ'P'$ may become an indefinitely thin slice, K will ultimately be in PQ, and k ultimately $= AM$ or x.

Also in this case $\dfrac{m'h' - mh}{m' - m}$, which is the ratio of the *increments* of mh and of m, will, by the principles of the differential calculus, ultimately become the ratio of their *differentials*. Hence taking the ultimate limits on both sides, which will necessarily be equal, we have

$$x = \frac{d \,.\, mh}{dm}.$$

$$\text{Hence, } \frac{d \,.\, mh}{dx} = \frac{d \,.\, mh}{dm}\,\frac{dm}{dx} = x\,\frac{dm}{dx};$$

$$\text{and integrating in } x, \; mh = \textstyle\int_x x\,\dfrac{dm}{dx};$$

$$\therefore\; h = \frac{\int_x x\,\dfrac{dm}{dx}}{m}.$$

We may thus find the distance of the center of gravity from the known plane Ay.

If Ax, perpendicular to Ay, be a line along which abscissas are measured; and if $AM = x$, $MP = y$, the curve may be defined by a relation between x and y, if the body be a plane figure, or a figure of revolution round Ax, and hence $\frac{dm}{dx}$ may be found.

In other cases we may suppose two planes, at right angles to each other, passing through Ax; and if y and z be the distances of P from these planes, the surface of the body may be defined by an equation between x, y, and z, whence dm may be found.

If the body be symmetrical on the two sides of Ax, supposing it to lie in a plane; or if its section by every plane passing through Ax be symmetrical to Ax, supposing it extended in three dimensions; its center of gravity will be in the line Ax: and hence, to determine its position, it is sufficient to find the value of GH.

If the body be not thus symmetrical with respect to Ax, its center of gravity will not necessarily be in that line. In this case it will be necessary, if the body lie in a plane, to find the distance of the center of gravity from some other line besides Ay; for instance, to find GF its distance from Ax: this may be found in the same way as GH. If the body have three dimensions, it will be necessary to find, by similar methods, the distances of the center of gravity from three known planes, which will determine its position.

We shall consider the cases separately.

1. *A Symmetrical Area.*

34. Let PAp, fig. 110, be a curvilinear area symmetrical with regard to AM. We may suppose it to be a lamina of matter whose thickness may be neglected; or, if the thickness be supposed finite and constant, the position of G will be the same. It is manifest that the weight or quantity of matter of any part, supposing the density uniform, will be as the magnitude, or as the area of that part, and may be represented by the area.

Now if the abscissa AM (fig. 110,) $= x$, and the ordinate $MP = y$, $\frac{dm}{dx}$ is the differential of the area PAp, or $2y$; hence, if $GA = h$, we have, by Art. 33,

$$h = \frac{\int_x 2yx}{\int_x 2y} = \frac{\int_x xy}{\int_x y}.$$

We shall give some instances of the application of this formula.

Ex. 1. The equation to a curve is $y = \frac{x(a^2 - x^2)^{\frac{1}{2}}}{a}$; to find the center of gravity of its area,

$$\int_x y = \int_x \frac{x(a^2 - x^2)^{\frac{1}{2}}}{a} = C - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3a};$$

C being an arbitrary constant; and if we suppose the area to be taken from the point where x and $y = 0$,

$$\int_x y = \frac{a^3 - (a^2 - x^2)^{\frac{3}{2}}}{3a}.$$

$$\int_x xy = \int_x \frac{x^2(a^2 - x^2)^{\frac{1}{2}}}{a} = \frac{\int_x x \,.\, x \,.\, (a^2 - x^2)^{\frac{1}{2}}}{a}.$$

But $\int_x x \,.\, x(a^2 - x^2)^{\frac{1}{2}} = -\frac{x \,.\, (a^2 - x^2)^{\frac{3}{2}}}{3} + \int_x \frac{(a^2 - x^2)^{\frac{3}{2}}}{3}$ *

$$= -\frac{x \,.\, (a^2 - x^2)^{\frac{3}{2}}}{3} + \int_x \frac{a^2 \,.\, (a^2 - x^2)^{\frac{1}{2}}}{3} - \int_x \frac{x^2 \,.\, (a^2 - x^2)^{\frac{1}{2}}}{3};$$

$$\therefore\ 4\int_x x^2(a^2 - x^2)^{\frac{1}{2}} = -x \,.\, (a^2 - x^2)^{\frac{3}{2}} + a^2 \,.\, \int_x (a^2 - x^2)^{\frac{1}{2}}$$

$$= -x(a^2 - x^2)^{\frac{3}{2}} + \frac{a^2}{2} \,.\, \left\{x \,.\, (a^2 - x^2)^{\frac{1}{2}} + a^2 \,.\, \text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C;$$

---

* By the formula $\int_x u\frac{dv}{dx} = uv - \int_x v\frac{du}{dx}$; see Lacroix's Elementary Treatise, &c. No. 148.

$$\therefore \int_x \frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{a}$$

$$= \frac{1}{8a} \cdot \left\{-2x(a^2 - x^2)^{\frac{3}{2}} + a^2 x \cdot (a^2 - x^2)^{\frac{1}{2}} + a^4 \cdot \text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C$$

$$= \frac{1}{8a} \cdot \left\{(a^2 - x^2)^{\frac{1}{2}} \cdot x \left[a^2 - 2(a^2 - x^2)\right] + a^4 \cdot \text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C$$

$$= \frac{1}{8a} \cdot \left\{(a^2 - x^2)^{\frac{1}{2}} \cdot (2x^2 - a^2) \cdot x + a^4 \cdot \text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C:$$

and the integral being taken from $x = 0$, and $y = 0$, we have $C = 0$.

$$\text{Hence } h = \frac{3}{8} \cdot \frac{(a^2 - x^2)^{\frac{1}{2}} \cdot (2x^2 - a^2) \cdot x + a^4 \cdot \text{arc}\left(\sin. = \frac{x}{a}\right)}{a^3 - (a^2 - x^2)^{\frac{3}{2}}}.$$

If we take the whole curve, that is, make $x = a$, we have

$$h = \frac{3}{8} \cdot \frac{a^4 \cdot \frac{\pi}{2}}{a^5} = \frac{3\pi}{16} \cdot a.$$

Similarly, we should obtain the following results:

Ex. 2. If PAp, fig. 110, be the common parabola,

$$AG = \frac{3}{5} AM.$$

Ex. 3. If PAp be any parabola whose equation is $y^{m+n} = a^m x^n$,

$$AG = \frac{m + 2n}{2m + 3n} AM.$$

Ex. 4. If PAp be a segment of a circle whose center is C,

$$CG = \frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 5. If BAb be a semi-circle; center C; (fig. 110.)

$$CG = \frac{4}{3\pi} AC.$$

Ex. 6. If PAp be any segment of an ellipse, whose semi-axes are $CA = a$, and $CB = b$;

$$CG = \frac{a^2}{b^2} \cdot \frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 7. If BAb be a semi-ellipse, with center C;

$$CG = \frac{4}{3\pi} AC.$$

Ex. 8. If PAp, fig. 112, be any segment of a hyperbola whose semi-axes are $CA = a$, $CB = b$;

$$CG = \frac{a^2}{b^2} \cdot \frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 9. If $P'Bbp'$ be a segment of the area contained between the two hyperbolas which are conjugate to PAp;

$$CG' - \frac{a^2}{b^2} \cdot \frac{P'M'^3 - CB^3}{3 P'Bbp'}.$$

Ex. 10. If BP' be a rectangular hyperbola whose asymptotes are CE and Ce, and if we complete the parallelogram $P'O$, we have for the area $PP'Q'Q$;

$$Cg = \frac{\text{area } P'O}{\text{area } P'Q'} \cdot mn.$$

Ex. 11. If PAp, fig. 110, be a cycloid; axis AC; we have for the whole cycloid,

$$AG = \frac{7}{12} AC.$$

Ex. 12. If $PApC$ be a sector of a circle; center C;

$$CG = \frac{2}{3} \cdot \frac{AC \cdot Pp}{\text{arc } Pp} = \frac{2}{3} \cdot \frac{\text{rad. chord}}{\text{arc}}.$$

2. *A Curvilinear Area not symmetrical.*

35. Let $BCQP$, fig. 111, be a curvilinear area bounded by two curves BP, CQ, and their ordinates. Let G be its

center of gravity, and GK, GH the co-ordinates of this point parallel to the known lines Ax and Ay. Then we may find GH as before, by the formula

$$h = \frac{\int_x x \frac{dm}{dx}}{m},$$

where the value of $\frac{dm}{dx}$ is the differential coefficient of the area $BCQP$.

If $AM = x$, $MP = y$, $MQ = y'$, we have $\frac{dm}{dx} = y - y'$;

$$\therefore\ h = \frac{\int_x (y - y')\, x}{\int_x (y - y')} \ldots\ldots\ldots\ldots\ldots (1).$$

But if we take the integral of 1 with respect to y, supposing the integral to begin when y is y', it will be $y - y'$: or $y - y' = \int_y 1$. Hence we have for h,

$$h = \frac{\int_x x \int_y 1}{\int_x \int_y 1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When an integral is to be found by two integrations, thus indicated by the double sign $\int_x \int_y$, the first integration is to be performed considering y as the variable quantity, and x as constant. We must then substitute for y the value which it has as a function of x, according to the manner in which the integral is limited, and must integrate the resulting expression considering x as the variable quantity.

Now if we perform the integrations with respect to y and x in a reverse order, we shall evidently obtain the same result.

Hence, instead of $\int_x x \int_y 1$, we may use $\int_y \int_x x$; and $\int_y \int_x 1$ instead of $\int_x \int_y 1$.

$$\text{But } \int_y \int_x x = \int_y \frac{x^2}{2} + C$$

$$= \int_y \frac{x^2 - x'^2}{2}$$

where x' is the value of x when the area begins, and x the

value where it ends, corresponding to a constant value of y. Hence

$$h = \frac{\int_y (x^2 - x'^2)}{2\int_y\int_x 1} \ldots\ldots\ldots (3).$$

In the same manner if k be the ordinate GK, we shall have

$$k = \frac{\int_x (y^2 - y'^2)}{2\int_x\int_y 1} \ldots\ldots\ldots (4).$$

where y' and y are the first and last values of the ordinate corresponding to a given value of x.

The value of k may be found by formulæ corresponding to any of those which we have obtained for h: and by taking a value of h and a value of k, we determine the position of the center of gravity.

Thus we may take formulæ (1) and (4), putting $\int_x (y - y')$ for $\int_x\int_y 1$.

$$\left.\begin{aligned} h &= \frac{\int_x (y - y')x}{\int_x (y - y')} \\ k &= \frac{\int_x (y^2 - y'^2)}{2\int_x (y - y')} \end{aligned}\right\} \ldots\ldots\ldots (5).$$

In these formulæ the value of y in terms of x is to be substituted, after which the integration is to be performed with respect to x, and to be taken between the limits corresponding to the extremities of the curve.

If the curvilinear space be bounded, as ADB, fig. 113, by the abscissa, the ordinate and the curve, we shall have $y' = 0$.

$$h = \frac{\int_x y x}{\int_x y}; \quad k = \frac{\int_x y^2}{2\int_x y} \ldots\ldots\ldots (6).$$

If the curvilinear space be bounded, as ACD, fig. 114, by the lines Ax, Ay, and by the curve, we might use the formulæ

$$h = \frac{\int_x xy}{\int_x y}; \quad k = \frac{\int_y y x}{\int_y x} \ldots\ldots\ldots (7).$$

the integration in the first beginning when $x = 0$, and in the second when $y = 0$.

Ex. 13. Let AB, fig. 113, be a parabola whose axis is AD: to find the center of gravity of the space ADB.

Let $y^2 = cx$; and by formula (6),

$$h = \frac{\int_x yx}{\int_x y}; \quad k = \frac{\int_x y^2}{2\int_x y}.$$

$$\text{Now } \int_x y = \int_x c^{\frac{1}{2}} x^{\frac{1}{2}} = \tfrac{2}{3} c^{\frac{1}{2}} x^{\frac{3}{2}},$$

$$\int_x yx = \int_x c^{\frac{1}{2}} x^{\frac{3}{2}} = \tfrac{2}{5} c^{\frac{1}{2}} x^{\frac{5}{2}},$$

$$\int_x y^2 = \int_x cx \quad = \tfrac{1}{2} c x^2;$$

$$\therefore\ h = \tfrac{3}{5} x, \ k = \tfrac{3}{8} c^{\frac{1}{2}} x^{\frac{1}{2}} = \tfrac{3}{8} y.$$

Hence, if we take $AH = \frac{3}{5} AD$, and $AK = \frac{3}{8} AC$, by completing the parallelogram, we have G the center of gravity of ABD.

3. *A Solid of Revolution.*

36. In a solid of revolution, whose axis is Ax, $\dfrac{dm}{dx} = \pi y^2$; hence

$$h = \frac{\int_x \pi y^2 x}{\int_x \pi y^2} = \frac{\int_x y^2 x}{\int_x y^2}.$$

And the center of gravity will be in the axis Ax: hence if we measure this value of h along Ax, we have the center of gravity.

Ex. 14. Let PAp, fig. 110, be a segment of a sphere whose center is C; $AC = a$, $AM = x$, $AG = h$,

$$h = \frac{8ax - 3x^2}{4(3a - x)}.$$

And for the whole hemisphere, when $x = a$,

$$h = \frac{5a}{8}.$$

Ex. 15. Let the body be a segment of a spheroid generated by the revolution of an elliptical segment PAp; the center of gravity will be the same as that of a segment of a sphere with the same axis and center C; or, as before,

$$h = \frac{8ax - 3x^2}{4(3a - x)}.$$

And for the hemispheroid, $h = \dfrac{5a}{8}$.

Ex. 16. Let PAp, fig. 112, be a hyperboloid with center C; $CA = a$, $AM = x$, $AG = h$;

$$h = \frac{8ax - 3x^2}{4(3a + x)}.$$

As x becomes very great, the value to which this tends is

$$h = \frac{3x}{4};$$

which agrees with the expression for a cone.

Ex. 17. If PAp be a paraboloid;

$$h = \frac{2x}{3}.$$

Ex. 18. If the figure be a frustum of a paraboloid, of which the radii of the less and of the greater ends are a and b, and the length of the axis x, the distance of the center from the smaller end, h;

$$h = \frac{a^2 + 2b^2}{a^2 + b^2} \cdot \frac{x}{3}.$$

Ex. 19. If PAp be a solid generated by the revolution of any parabola whose equation is

$$y^{m+n} = a^m x^n;$$

$$h = \frac{m + 3n}{2m + 4n} \cdot x.$$

4. *Any Solid.*

37. Let PBQ, fig. 115, represent ony solid bounded by a surface to which we have an equation in terms of three rectangular co-ordinates x, y, z. Let Ax be the direction of one of the co-ordinates, and let the body h be cut by a plane PM perpendicular to Ax. Let A be the area of the section of the body made by this plane. Then $\frac{dm}{dx}$ will $= A$.

Now the boundaries of the plane A perpendicular to AM will be determined by the co-ordinates y and z, which are perpendicular to AM, and in the plane A. Hence we shall have

$$A = \int_y z, \text{ or as before } A = \int_y \int_z 1. \quad \text{And } \frac{dm}{dx} = \int_y \int_z 1;$$

$$\therefore h = \frac{\int_x x \int_y \int_z 1}{\int_x \int_y \int_z 1}.$$

Or, since, as in Art. 35, the order of the integrations is indifferent*,

$$h = \frac{\int_x \int_y \int_z x}{\int_x \int_y \int_z 1};$$

$$\text{similarly, } k = \frac{\int_x \int_y \int_z y}{\int_x \int_y \int_z 1},$$

$$l = \frac{\int_x \int_y \int_z z}{\int_x \int_y \int_z 1}:$$

k, l being the co-ordinates of the center of gravity parallel respectively to y and to z.

If we suppose the integration in z to be performed, we shall have

---

* The expression for the solidity of a body is $\int_x \int_y z$. Similarly it is $\int_y \int_z x$, and $\int_z \int_x y$. The expression $\int_x \int_y \int_z 1$ comprehends all these three. For the order of the integrations is indifferent as in p. 46; and if we make the first integration with respect to z, we obtain $\int_x \int_y z$: if with respect to x, we have $\int_y \int_z x$; if with respect to y, we have $\int_z \int_x 1$. Similarly $\int_x \int_y \int_z 1$ is the same as $\int_x \int_y \int_z x$; and so of the rest.

$$h = \frac{\int_x \int_y xz}{\int_x \int_y z}.$$

$$k = \frac{\int_x \int_y yz}{\int_x \int_y z},$$

$$l = \frac{\int_x \int_y z^2}{2\int_x \int_y z}.$$

And z being known in terms of x, y, its value may be bstituted, and the integrations in y performed, between the 'oper limits. Then the value of y in terms of x may be bstituted; and the integrations performed with respect to will give the value of h.

Ex. 20. Let the body be a fourth part of a paraboloid revolution; as $ABCD$, fig. 115; cut off by a plane BAC erpendicular to the axis, and by two planes BAD, CAD, erpendicular to the preceding and to each other; to find its nter of gravity.

Let A be the origin, and AB, AC, AD the axes of the ctangular co-ordinates x, y, z, respectively. If $AM = x$, $N = MO = y$, $OP = z$, the equation to the surface will be

$$x^2 + y^2 + bz = a^2,$$

here AB or $AC = a$, and the axis $AD = \dfrac{a^2}{b}$,

$$\int_x \int_y z = \int_x \int_y \left(\frac{a^2 - x^2 - y^2}{b}\right) = \frac{1}{b}\int_x \int_y (a^2 - x^2 - y^2).$$

nd, integrating first for y,

$$= \frac{1}{b}\int_x \left(a^2 y - x^2 y - \frac{y^3}{3}\right).$$

The limits of the integration for y are determined by the ature of the part considered; if it is to be bounded by a lane RNO parallel to the plane of xz at a distance AN, must be taken from o to AN; and in the next integration nust be supposed constant. Hence we have

$$\int_x\int_y z = \frac{1}{b}\left(a^2xy - \frac{x^3y}{3} - \frac{y^3x}{3}\right).$$

Where the limits of the integration for x are determined in the same way as for y. If the solid be bounded by a plane QMO parallel to the plane of yz, at a distance $x = AM$, the quantity now found expresses the solid; or

$$\text{solid } AP = \frac{xy}{b}\left(a^2 - \frac{x^2+y^2}{3}\right) = \frac{xy\,.\,(3a^2-x^2-y^2)}{3b}.$$

If the solid be not bounded by a plane RNO, but continued till its surface meets the plane CAB in Cm, we must, after the integration for y, put for y the value which it assumes by making

$$z = 0, \text{ or } x^2+y^2 = a^2, \text{ whence } y^2 = a^2 - x^2.$$

Hence

$$\int_x\int_y z = \frac{1}{b}\int dx\left(a^2y - x^2y - \frac{y^3}{3}\right)$$

$$= \frac{1}{b}\int_x\left(a^2 - x^2 - \frac{y^2}{3}\right).y$$

$$= \frac{2}{3b}\int_x (a^2-x^2)^{\frac{3}{2}},$$

which (taken from $x = 0$) gives the solid $ACmQD$

$$= \frac{2}{3.8.b}\left\{2x(a^2-x^2)^{\frac{3}{2}} + 3a^2x(a^2-x^2)^{\frac{1}{2}} + 3a^4.\text{arc}\left(\sin. = \frac{x}{a}\right)\right\}.$$

And if we take the whole solid, x must be taken $= a$; in this case the arc will become $\frac{\pi}{2}$, and we shall have

$$\text{whole solid } ABCD = \frac{2}{3.8.b}.3a^4.\frac{\pi}{2} = \frac{\pi a^4}{8b}.$$

The solidity of the whole might be more simply found; for it will manifestly be $\frac{1}{4}$ of the whole paraboloid; and a

paraboloid is $\frac{1}{2}$ the cylinder on the same base (rad. $= a$), and with the same altitude; hence

$$\text{whole solid} = \frac{1}{4} \cdot \frac{1}{2} \pi a^2 . \frac{a^2}{b} = \frac{\pi a^4}{8b}, \text{ as before.}$$

We now proceed to find $\int_x \int_y x z$,

$$\int_x \int_y x z = \int_x \int_y x \left(\frac{a^2 - x^2 - y^2}{b} \right)$$

$$= \frac{1}{b} \int_x x \int_y (a^2 - x^2 - y^2)$$

$$= \frac{1}{b} \int_x x \left(a^2 y - x^2 y - \frac{y^3}{3} \right);$$

the limits of y as before. On the first supposition, that the body is bounded by planes RNO, QMO, we have

$$\int_x \int_y x z = \frac{1}{b} \left(\frac{a^2 x^2 y}{2} - \frac{x^4 y}{4} - \frac{x^2 y^3}{6} \right)$$

$$= \frac{x^2 y (6a^2 - 3x^2 - 2y^2)}{12b};$$

hence, for AP,

$$h = \frac{b \left\{ x^2 y (6a^2 - 3x^2 - 2y^2) \right\}}{4b \left\{ xy (3a^2 - x^2 - y^2) \right\}}$$

$$= \frac{x}{4} \cdot \frac{6a^2 - 3x^2 - 2y^2}{3a^2 - x^2 - y^2}.$$

If we suppose $AMON$, the base of the figure, to be a square, or $y = x$; this becomes

$$h = \frac{x}{4} \cdot \frac{6a^2 - 5x^2}{3a^2 - 2x^2}.$$

If we suppose the quadrilateral curve surface $DQPR$ to have its angle P in the circumference of the base BC, as at m, we shall have $z = 0$; and hence

$x^2 + y^2 = a^2$, or $y^2 = a^2 - x^2$; and hence for the solid Mr,

$$h = \frac{x}{4} \cdot \frac{4a^2 - x^2}{2a^2}.$$

On the second supposition, that the surface of the solid is to be continued till it meets the plane ABC, we must, after the integration for y, substitute for y its value in that plane, that is, $y = \sqrt{(a^2 - x^2)}$; hence we have

$$\int_x \int_y xz = \frac{1}{b} \int x\,dx \left(a^2 y - x^2 y - \frac{y^3}{3}\right)$$

$$= \frac{1}{b} \int_x x \left(a^2 - x^2 - \frac{a^2 - x^2}{3}\right) . \sqrt{(a^2 - x^2)}$$

$$= \frac{2}{3b} \cdot \int_x x . (a^2 - x^2)^{\frac{3}{2}}$$

$$= \frac{2}{3b} \cdot \left(\frac{a^5}{5} - \frac{(a^2 - x^2)^{\frac{5}{2}}}{5}\right),$$

taking the integral from $x = 0$; and for the whole solid $ABCD$, or when $x = 0$, it becomes

$$= \frac{2}{3b} \cdot \frac{a^5}{5} = \frac{2a^5}{15b}.$$

Hence, for the whole solid $ABCD$,

$$h = \frac{2a^5}{15b} \cdot \frac{8b}{\pi a^4} = \frac{16a}{15\pi} = \frac{a}{3}, \text{ nearly.}$$

In the same way it might be shewn that, for the part of the solid bounded by planes parallel to the planes of xz and xy, we have

$$k = \frac{y}{4} \cdot \frac{6a^2 - 3y^2 - 2x^2}{3a^2 - y^2 - x^2};$$

$$l = \frac{3\left\{a^4 - \frac{2}{3} a^2 (x^2 + y^2) + \frac{2}{9} x^2 y^2 + \frac{1}{5}(x^4 + y^4)\right\}}{2b . (3a^2 - x^2 - y^2)}.$$

And for the whole solid,

$$k = \frac{16a}{15\pi},$$

$$l = \frac{b^2}{3a} = \frac{\text{axis}}{3},$$

ıich last result also follows from Ex. 17.

5. *A Plane Curve.*

38. When the body is a curve lying in one plane, if we ppose it to be a physical line of inconsiderable thickness, ds :ing the differential of its length, $\frac{dm}{dx}$ will be as $\frac{ds}{dx}$.

Hence,

$$h = \frac{\int_x x \frac{ds}{dx}}{\int_x \frac{ds}{dx}}.$$

But we have $\frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$ (Lacroix, *Elementary* *ʳreatise*, Art. 75.). Therefore

$$h = \frac{\int_x x \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int_x \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

$$\text{Similarly, } k = \frac{\int_x y \frac{ds}{dx}}{\int_x \frac{ds}{dx}};$$

$$\text{or, } k = \frac{\int_x y \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int_x \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

If the curve be symmetrical with respect to Ax, it will be sufficient to find h, since the center of gravity will be in Ax.

Ex. 21. Let PAp, fig. 110, be a circular arc, center C, radius $= a$.

Let arc $AP = s$; $\therefore CM = x = a \cdot \cos. \dfrac{s}{a}$;

$$\therefore \int_s x \frac{ds}{dx} = a \int_s \cos. \frac{s}{a} \cdot \frac{ds}{dx} = a^2 \sin. \frac{s}{a}.$$

And if the whole arc be $2l$, and its middle point A, the integral must be taken from

$$s = -l \text{ to } s = l; \therefore \int_s x \frac{ds}{dx} = 2a^2 . \sin. \frac{l}{a}.$$

$$\text{Hence } CG = h = \frac{2a^2 . \sin. \frac{l}{a}}{2l} = \frac{a . 2 \sin. l \,(\text{rad.} = a)}{2l}$$

$$= \frac{\text{radius . chord}}{\text{arc}}.$$

Cor. Hence for the semi-circle, $h = \dfrac{2a}{\pi}$.

Ex. 22. Let APB, fig. 113, be a semi-cycloid with axis AD.

$$AH = h = \frac{AD}{3}, \quad HG = k = DB - \frac{2}{3} AD.$$

Cor. Hence $CK = DH$.

Ex. 23. Let PAp, fig. 110, be a catenary of which A is the lowest point. Take AD vertical and equal to a length of the string equivalent to the tension; then

$$DG = \tfrac{1}{2} DM + \frac{DA . MP}{PAp}.$$

6. *A Curve of double Curvature.*

'. Let s be the length of the curve; and, as before, $\frac{ds}{dx}$. Then, if x, y, z be the rectangular co-ordinates curve,

$$\frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)}, \text{ and}$$

$$h = \frac{\int_x x \frac{ds}{dx}}{\int_x \frac{ds}{dx}}; \quad k = \frac{\int_x y \frac{ds}{dx}}{\int_x \frac{ds}{dx}}, \quad l = \frac{\int_x z \frac{ds}{dx}}{\int_x \frac{ds}{dx}}.$$

. 24. Let the curve be the thread of a screw, of which s is Az.

thread, projected on the plane xy, will become a circle; be the radius of this circle, $a^2 = x^2 + y^2$. Also $\frac{x}{a}$ will cosine of the arc of this circle, corresponding to any n the curve, and z will be proportional to this arc. he equations to the curve are

$$y = \sqrt{(a^2 - x^2)},$$

$$z = m \cdot \text{arc}\left(\cos. = \frac{x}{a}\right),$$

; a constant quantity, and the thread of the screw being d to begin in the line Ax.

$$\text{Hence } \frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}},$$

$$\frac{dz}{dx} = -\frac{m}{\sqrt{(a^2 - x^2)}};$$

$$\therefore \frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)}$$

$$= -a\frac{\sqrt{(a^2+m^2)}}{\sqrt{(a^2-x^2)}}.$$

Hence also $\dfrac{ds}{dx} = \dfrac{\sqrt{(a^2+m^2)}}{m}\dfrac{dz}{dx}$,

and $s = \dfrac{\sqrt{(a^2+m^2)}}{m}z$.

Now $\int_x x\dfrac{ds}{dx} = \int_x -x\dfrac{\sqrt{(a^2+m^2)}}{\sqrt{(a^2-x^2)}}$

$$= \sqrt{(a^2+m^2)}\cdot\sqrt{(a^2-x^2)};$$

which begins when $x = a$.

Also $\int_x \dfrac{ds}{dx} = \int_x -\sqrt{(a^2+m^2)}$

$$= (a-x)\sqrt{(a^2+m^2)},$$

which also begins when $x = a$.

And $\int_z z\dfrac{ds}{dx} = \int_z \dfrac{\sqrt{(a^2+m^2)}}{m}z\dfrac{dz}{dx}$

$$= \frac{\sqrt{(a^2+m^2)}}{m}\cdot\frac{z^2}{2}.$$

Hence

$$h = \frac{m\sqrt{(a^2-x^2)}}{z};$$

$$k = \frac{m(a-x)}{z};$$

$$l = \frac{z}{2}.$$

If $x = a$, that is, if the spiral consist of a complete number of revolutions, $h = 0$, $k = 0$. In this case the center of gravity is in the axis, and in the middle of its length.

$x = 0$, $h = \dfrac{ma}{z}$, $k = \dfrac{ma}{z}$: in this case the spiral con-
' a complete number of revolutions together with a
of a revolution.

7. *A Surface of Revolution.*

If s be the length of the curve, $2\pi y \dfrac{ds}{dx}$ is the differ-
coefficient of the surface with respect to x, and, as
this may be put for $\dfrac{dm}{dx}$. Also the center of gravity
in the axis of revolution. Hence

$$h = \frac{\int_x xy \frac{ds}{dx}}{\int_x y \frac{ds}{dx}} = \frac{\int_x yx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int_x y \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

. 25. If the surface be a cone, and h the distance from
tex,

$$h = \frac{2x}{3}.$$

. 26. If the surface be a sphere, and h measured from
:ex,

$$h = \frac{x}{2}.$$

8. *Any Surface.*

Let the surface be defined by any equation $u = 0$,
ı x, y, z; whence we may find z in terms of x and y.

t $\dfrac{dz}{dx} = p$, $\dfrac{dz}{dy} = q$; h, k, l as in Art. 37. The differ-

ential coefficient, taken with regard to x and y successively, of the surface, is $(1+p^2+q^2)^{\frac{1}{2}}$; hence as before,

$$h = \frac{\int_x \int_y x\,(1+p^2+q^2)^{\frac{1}{2}}}{\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}}};$$

$$k = \frac{\int_x \int_y y\,(1+p^2+q^2)^{\frac{1}{2}}}{\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}}};$$

$$l = \frac{\int_x \int_y z\,(1+p^2+q^2)^{\frac{1}{2}}}{\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}}}.$$

Ex. 27. Let a conical surface be divided into four parts by two planes perpendicular to each other, passing through the axis. To find the center of gravity of one of these parts: as BCD, fig. 115; DB and DC being supposed to be straight lines.

If we make the vertex D the origin of co-ordinates, the axis the line of z, and measure x, y, parallel to AB, AC, respectively, we have

$$z = m\sqrt{(x^2+y^2)};$$

Where m is the tangent of the angle which the slant side makes with the base.

Hence

$$p = \frac{dz}{dx} = \frac{mx}{\sqrt{(x^2+y^2)}},$$

$$q = \frac{dz}{dx} = \frac{my}{\sqrt{(x^2+y^2)}};$$

$$\therefore\ (1+p^2+q^2)^{\frac{1}{2}} = (1+m^2)^{\frac{1}{2}},$$

$$\therefore\ \int_x \int_y (1+p^2+q^2)^{\frac{1}{2}} = (1+m^2)^{\frac{1}{2}} \cdot \int_x y.$$

And if the curve surface be a quadrilateral figure $DQPR$ bounded by planes parallel to those of the co-ordinates,

$$\text{this} = (1+m^2)^{\frac{1}{2}}\, xy.$$

But if we take the surface as bounded by a plane BAC perpendicular to the axis at the distance $a = DA$, we must have, after the integration for y, $z = a$, the axis;

$$\text{Now } a^2 - m^2x^2 = m^2y^2,\ y = \frac{\sqrt{(a^2 - m^2x^2)}}{m},$$

$$\therefore \int_x\int_y (1 + p^2 + q^2)^{\frac{1}{2}} = \frac{(1 + m^2)^{\frac{1}{2}}}{m} \cdot \int_x (a^2 - m^2x^2)^{\frac{1}{2}}$$

$$= \frac{(1 + m^2)^{\frac{1}{2}}}{2m}\left\{(a^2 - m^2x^2)^{\frac{1}{2}} \cdot x + \frac{a^2}{m} \cdot \text{arc}\left(\sin. = \frac{mx}{a}\right)\right\} + \text{const.}$$

and, from $x = 0$ to $x = AB = \dfrac{a}{m}$

$$= \frac{(1 + m^2)^{\frac{1}{2}}}{2m} \cdot \frac{a^2 \cdot \pi}{2m} = \frac{(1 + m^2)^{\frac{1}{2}} \cdot \pi a^2}{4m^2},$$

which might be deduced also from the known method of finding the surface of a cone.

To find the numerator of h, we have it

$$= (1 + m^2)^{\frac{1}{2}} \int_x\int_y x = (1 + m^2)^{\frac{1}{2}} \int_x y x,$$

and, for the quadrilateral $DPQR$, $= \dfrac{(1 + m^2)^{\frac{1}{2}}}{2} y x^2$.

But, for the whole surface DBC,

$$= \frac{(1 + m^2)^{\frac{1}{2}}}{m} \int_x (a^2 - m^2x^2)^{\frac{1}{2}} x$$

$$= -\frac{(1 + m^2)^{\frac{1}{2}}}{3m^3} (a^2 - m^2x^2)^{\frac{3}{2}} + \text{constant}:$$

and, taken from $x = 0$ to $x = AB = \dfrac{a}{m}$

$$= \frac{(1 + m^2)^{\frac{1}{2}} a^3}{3m^3}.$$

The numerator of k will manifestly be the same.

Similarly, for the numerator of l,

$$\int_x\int_y(1+p^2+q^2)^{\frac{1}{2}}z = (1+m^2)^{\frac{1}{2}}\int_x\int_y z$$
$$= (1+m^2)^{\frac{1}{2}}\int_x\int_y m\,(x^2+y^2)^{\frac{1}{2}}$$
$$= (1+m^2)^{\frac{1}{2}}\,.\,m\,.\int_x\left\{\frac{(x^2+y^2)^{\frac{1}{2}}y}{2}+\frac{x^2}{2}\,\mathrm{l}\,\frac{y+\sqrt{(x^2+y^2)}}{x}\right\},$$

in which the denominator x is given to the quantity under the logarithmic sign that the integral may begin when $y=0$. For the quadrilateral surface $DPQR$, we must now integrate for x, supposing y constant; and the double integral becomes

$$= \frac{(1+m^2)^{\frac{1}{2}}\,.\,m}{6}\left\{2xy\,.\,(x^2+y^2)^{\frac{1}{2}}+x^3\,\mathrm{l}\,\frac{y+\sqrt{(x^2+y^2)}}{x}\right.$$
$$\left.+\,y^3\,\mathrm{l}\,\frac{x+\sqrt{(x^2+y^2)}}{y}\right\}.$$

For the whole surface DBC, we must, after integrating for y, put for y the value $\dfrac{\sqrt{(a^2-m^2x^2)}}{m}$; and it becomes,

$$= \frac{(2+m^2)^{\frac{1}{2}}\,.\,m}{2}\int_x\left\{\frac{a\sqrt{(a^2-m^2x^2)}}{m^2}+x^2\,\mathrm{l}\,\frac{\sqrt{(a^2-m^2x^2)}+a}{mx}\right\};$$

and the integral being taken from $x=0$ to $x=\dfrac{a}{m}$; this becomes

$$= \frac{(1+m^2)^{\frac{1}{2}}\,.\,\pi a^3}{6m^2}.$$

Hen̆ce, for the whole conical surface,

$$h = k = \frac{(1+m^2)^{\frac{1}{2}}\,.\,a^3}{3m^3}\cdot\frac{4m^2}{(1+m^2)^{\frac{1}{2}}\,.\,\pi a^2} = \frac{4a}{3\pi m},$$

$$l = \frac{(1+m^2)^{\frac{1}{2}}\,.\,\pi a^3}{6m^2}\cdot\frac{4m^2}{(1+m^2)^{\frac{1}{2}}\,.\,\pi a^2} = \frac{2a}{3},$$

which last agrees with Ex. 33, as it should.

For the center of gravity of the quadrilateral surface $DPQB$, where $AM=x$, $AN=y$, we have for h, k, l, other expressions which are easily deduced from the results given above.

9. *Guldinus's Properties**.

?. PROP. *If any plane figure revolve about an axis own plane, the content of the solid generated by this in its revolution is equal to a prism whose base is evolving figure, and its height the length of the path bed by the center of gravity of the plane figure.*

he figure may either be bounded by straight lines, or ; or by a combination of the two; and the revolution ake place either through a whole circumference or any of it.

Ve shall suppose the whole of the revolving figure to one side of the axis.

et AB, fig. 116, be the axis of revolution, PQR the ; G its center of gravity; GK, PQM, ordinates per-ular to AB. And let the figure revolve into the on $P'Q'R'$; the angle PMP' being $=\theta$. Also let x, $PM=y$, $MQ=y'$, $GK=k$.

he sector $PMP'=\frac{1}{2}y^2\theta$, and $QMQ'=\frac{1}{2}y'^2\theta$. Hence

$$PQQ'P'=\tfrac{1}{2}(y^2-y'^2)\,\theta;$$

ence the small increment of the solid PR' corresponding is $\frac{1}{2}(y^2-y'^2)\,\theta\,\delta x$. Hence the solid

$$=\frac{\theta}{2}\int_x(y^2-y'^2).$$

lso the prism whose base is PQR, and altitude the arc is $=PQR\,.\,GG'$; and area $PQR=\int_x(y-y')$, $GG'=k\theta$. this prism $=k\theta\int_x(y-y')$.

ut by formula (5) for k, in Art. 35, we have

$$k\int_x(y-y')=\tfrac{1}{2}\int_x(y^2-y'^2).$$

---

e propositions known by this name were discovered by Pappus, and re-d about 1640, by Guldin or Guldinus, a Jesuit, who was Professor of atics at Rome.

Therefore the figure described by the revolution of PQR is equal to the prism mentioned in the Proposition.

If the figure be composed of several curves, or of straight lines, both the numerator and denominator of k will consist of several integrals added together, corresponding to the different parts of the figure. Also both the area of the figure, and the content of the solid will consist of parts corresponding to these; and the solid and the prism will be found to be equal in the same manner as before.

43. PROP. *If any plane figure revolve about any axis in its own plane, the area of the surface generated by the perimeter of this figure in its revolution is equal to a rectangle, one of whose sides is the perimeter, and the other the length of the path described by the center of gravity of the perimeter.*

The denominations remaining as in last Article, let δs be the small increment of the length of the curve corresponding to δx; and since $y\theta$ is the length of the path described by P, $y\theta \, . \, \delta s$ is the increment of the surface described by the revolution; and $\theta \int_x y \frac{ds}{dx}$ is the whole surface.

Also the whole perimeter is $\int_x \frac{ds}{dx}$; and if G be now its center of gravity of the perimeter, $k\theta$ is the path described by the center of gravity of the perimeter; and $k\theta \int_x \frac{ds}{dx}$ is the rectangle mentioned in the proposition.

But by Art. 38,

$$k \int_x \frac{ds}{dx} = \int_x y \frac{ds}{dx},$$

whence the proposition is manifest.

44. Hence we may find the contents and areas of surfaces of revolution whenever we can find the area or perimeter of the revolving figure and its center of gravity.

Ex. 28. Let the figure be a circle which, revolving round an axis without it, generates a solid, resembling a cylinder

bent so as to return into itself, or the ring of an anchor. The center of the circle will be the center of gravity both of the area and of the perimeter. Hence, by Article 42, the solid content of such a ring is equal to the cylinder whose base is the revolving circle and its length the circle described by the center of the circle. Also by Article 43, the surface of the ring is equal to the rectangle contained by the circumference of the revolving circle and the path of its center; that is, it is equal to the surface of the cylinder beforementioned. Hence if we could suppose that the ring was cut through in some part, and unrolled into a cylinder so that its axis should remain of the same length as before, both the solidity and the surface would continue unaltered.

CHAP. V.

THE EQUILIBRIUM OF A FLEXIBLE BODY.

45. The equilibrium of a flexible body depends upon the same conditions as that of a rigid one, and may be deduced from the principles already laid down. These principles may be applied by observing that, in all cases, *a flexible body may be supposed to become rigid after the equilibrium is established:* or, that the forces which keep a flexible body at rest would keep at rest a rigid body of the same form. For after the body has assumed the form which the forces produce in it, there is no tendency to change the form; hence it makes no difference whether we suppose the body to have the property of resisting a change of form or not: that is, it makes no difference whether we suppose the body to be rigid or flexible.

We shall suppose the bodies to be *perfectly* flexible, that is, to offer *no* resistance to any change of figure.

The *Tension* of a string or chain is the force exerted by one part upon another contiguous part in the direction of its length. Every point of the string must be acted upon by equal and opposite forces of this kind: and a force of the same kind is exerted upon any fixed point to which the string is attached.

We shall consider the equilibrium of a flexible *line**, acted on by various forces. This line may be supposed to be a *cord*, indefinitely slender and perfectly void of stiffness; or a *chain* composed of indefinitely small links. On this account the curve formed by the line is called the *Catenary*.

1. *The Catenary, when a uniform Chain is acted on by Gravity.*

46. Prop. *To find the equation to the catenary between* x Z *and* s, *beginning at the lowest point.*

Let AB, fig. 117, represent the catenary. Let C be the lowest point, CM vertical $= x$, MP horizontal $= y$, $CP = s$. The portion CP may be supposed to become rigid after it has assumed the form of equilibrium; and since its weight and figure remain the same as before, it will be supported in the same manner. Now the forces which act upon the portion CP are, besides its own gravity, the tension at C and the tension at P: and these three forces must keep CP in equilibrium. Also the tensions are in the directions of the tangents RC and RP at C and P.

Let PR meet MC in T, PM will be parallel to RC; and hence the three lines MT, PM, TP are parallel to the directions of the three forces (gravity, tension at C, tension at P), which keep CP at rest, and hence (*Elem. Tr.*) the forces will be as those three lines. Hence

$$\frac{\text{tension at } C}{\text{weight of } CP} = \frac{MP}{TM}.$$

* Flexible bodies may be lines, surfaces, or solids. A flexible *line* can in all cases be extended into a straight line. A flexible surface is not necessarily susceptible of being unrolled into a plane, without stretching or tearing; if it be capable of this, it is called a *developable surface*.

Let the tension of the string at C be equal to the weight of a length c of the string; the weight of the length CP will be as CP or s; and the first side of the above equation will be $\frac{c}{s}$. Also $\frac{dy}{dx}$ will be equal to the second side.

$$\text{Hence } \frac{c}{s} = \frac{dy}{dx} \quad \ldots\ldots\ldots\ldots (1);$$

from which equation the properties of the curve may be deduced.

If we square both sides of equation (1) and add unity to them, we have

$$\frac{c^2 + s^2}{s^2} = 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2;$$

$$\therefore \frac{dx}{ds} = \frac{s}{\sqrt{(c^2 + s^2)}} \quad \ldots\ldots (2),$$

and integrating with respect to s, supposing $s = 0$ when $x = 0$,

$$x + c = \sqrt{(c^2 + s^2)} \quad \ldots\ldots\ldots (3).$$

Hence also we find

$$s = \sqrt{(x^2 + 2cx)} \quad \ldots\ldots (4).$$

Cor. 1. If the angle which the curve makes with the vertical be called α, we have

$$\tan. \alpha = \frac{dy}{dx} = \frac{c}{s};$$

$$\cos. \alpha = \frac{dx}{ds} = \frac{s}{\sqrt{(c^2 + s^2)}}.$$

Cor. 2. For the tension at any point P,

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{TP}{TM} = \frac{ds}{dx} = \frac{\sqrt{(c^2 + s^2)}}{s};$$

and weight of $CP = s$; $\therefore$ tension at $P = \sqrt{(c^2 + s^2)} = x + c$.

Hence, and from the last Corollary, it appears that

$$\text{tension at } P = \frac{s}{\cos. \alpha}.$$

COR. 3. If we take $CD = c$, and draw DQ horizontal, and PQ vertical, $PQ = DM = x + c =$ tension at P.

COR. 4. If we put $DM = u$, $x = u - a$, and hence

$$s = \sqrt{(u^2 - a^2)},$$
$$u = \sqrt{(s^2 + a^2)}.$$

47. PROP. *To find the equation between* y *and* s.

As in last Article,

$$\frac{s}{c} = \frac{dx}{dy} \text{ .... (1)}$$

$$\frac{s^2 + c^2}{c^2} = 1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{ds}{dy}\right)^2;$$

$$\frac{1}{c} \cdot \frac{dy}{ds} = \frac{1}{\sqrt{(s^2 + c^2)}} \text{ .... (2).}$$

Integrating with respect to s, supposing $y = 0$ when $s = 0$,

$$\frac{y}{c} = \text{l} \frac{s + \sqrt{(s^2 + c^2)}}{c} \text{ ...... (3).}$$

$$\text{Hence, } \epsilon^{\frac{y}{c}} = \frac{s + \sqrt{(s^2 + c^2)}}{c},$$

$$\therefore \epsilon^{-\frac{y}{c}} = \frac{c}{\sqrt{(s^2 + c^2)} + s} = \frac{\sqrt{(s^2 + c^2)} - s}{c}.$$

Subtracting, and reducing

$$s = \frac{c}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{\frac{y}{c}}\right) \text{ .... (4).}$$

COR. We have

$$\frac{\text{tension at } P}{\text{tension at } C} = \frac{TP}{TM} = \frac{ds}{dy} = \frac{\sqrt{(s^2 + c^2)}}{c}.$$

Also as before, tension at $P = \dfrac{c}{\sin. \alpha}$.

48. Prop. *To find the equation between* x *and* y.

If in the equation (1) of last Article we put the value of s rom (4), we have

$$\frac{dx}{dy} = \frac{1}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{-\frac{y}{c}}\right) \text{............} (1).$$

Integrate with respect to y ($x = 0$ when $y = 0$),

$$x + c = \frac{c}{2}\left(\epsilon^{\frac{y}{c}} + \epsilon^{-\frac{y}{c}}\right) \text{..............} (2).$$

Again, if in equation (1) of Art. 46, we put for s its lue from (4) of that Article, we have

$$\frac{dy}{dx} = \frac{c}{\sqrt{(x^2 + 2cx)}} \text{................} (3).$$

Integrate with respect to x,

$$\therefore y = c \, \mathrm{l} \frac{x + c + \sqrt{(x^2 + 2cx)}}{c} \text{......} (4).$$

49. Prop. *To find the equations to the catenary beinning from any point.*

Let A, fig. 118, be a point which is considered as the eginning of the catenary, AP any arc. Let the curve of quilibrium be continued if necessary, and let C be its lowest oint. Let AN vertical $= x$, NP horizontal $= y$, $AP = s$.

The portion AP will be kept in equilibrium in the same orm whether we suppose it to be acted on at A by the tension of CA, or by the re-action of a fixed point. But if we suppose AP to be a portion of CAP, its form will be determined as in the preceding Articles. Let c be the tension at the lowest point C, $CP = s'$, and we have as before,

$$\frac{c}{s'} = \frac{dy}{dx} \text{.......................} (1),$$

for $\frac{dy}{dx}$ is the same whether x and y be CM, MP, or AN, NP.

$$= \mathrm{l}\left\{\frac{dx}{dy} + \frac{ds}{dy}\right\}$$

$$= \mathrm{l}\left\{\frac{\cos.\alpha}{\sin.\alpha} + \frac{1}{\sin.\alpha}\right\}.$$

$$\text{Also } c = s\frac{dy}{dx} = l\tan.\alpha, \text{ and } y = h;$$

$$\therefore h = l\tan.\alpha\,\mathrm{l}\,\frac{1+\cos.\alpha}{\sin.\alpha} = \tan.\alpha\,\mathrm{l}\,\text{co-tan.}\frac{\alpha}{2};$$

$$\therefore \frac{h}{l} = -\tan.\alpha\,.\,\mathrm{l}\tan.\frac{\alpha}{2}.$$

From this equation we have to determine α. This cannot be done directly, but it is easy to approximate to it with sufficient rapidity. For this purpose it will be proper to adapt the formula last found, which is calculated with Napierian logarithms and a radius $= 1$, to the common tables. Let Tan. be the tangent and R the radius of the tables; the Napierian logarithm of $10 = 2.3025851 = M$; and making log. signify the logarithm to base 10, we have

$$\frac{h}{l} = -\frac{\text{Tan.}\,\alpha}{R}.M.\log.\frac{\text{Tan.}\frac{\alpha}{2}}{R};$$

$$= -\frac{\text{Tan.}\,\alpha}{R}.M.\log.\frac{R}{\text{Co-tan.}\frac{\alpha}{2}}$$

$$= \frac{\text{Tan.}\,\alpha}{R}.M.\left(\log.\text{Co-tan.}\frac{\alpha}{2} - \log.R\right);$$

$$\therefore \log.\frac{h}{l} = \log.\text{Tan.}\alpha + \log.\left(\log.\text{Co-tan.}\frac{\alpha}{2} - \log.R\right) + \log.M - \log.R,$$

where $\log.R = 10$, $\log.M = .3622157$.

Assuming values of α, we may calculate $\frac{h}{l}$, and by observing the error of the result obtain a more accurate value of α.

Ex. Let the string $BCB' = 2BB'$, to find the position.

We have $\log.\frac{h}{l} = \log.\frac{1}{2} = \bar{1}.6989700.$

By a few trials we shall find that $\alpha = 13^0$ will nearly give this value by the formula.

13^0 would give $\log.\frac{h}{l} = \bar{1}.7002484$; therefore 13^0 is too large:

$12^0.30'$............$\log.\frac{h}{l} = \bar{1}.6904752$; therefore $12^0.30'$ is too small.

Hence, since the differences of the results, when very small, are nearly proportional to the differences of the suppositions;

$$7002484 - 6904752 : 7002484 - 6989700 :: 30' : 4', \text{ nearly.}$$

Therefore $\alpha = 13^0 - 4' = 12^0.\ 56'$ very nearly; and by repeating the process we might obtain the value of α still more accurately.

Knowing α, we know $a = \frac{l}{\cos.\alpha} = l \sec.\alpha.$

To find the depth EC, to which the vertex hangs, we have, by Art. 49,

$$EC = BF - CD = a - a\sin.\alpha = l.(\sec.\alpha - \tan.\alpha).$$

For c, the tension at the point C, we have, by the same Article,

$$c = a\sin.\alpha = l\tan.\alpha.$$

In the case just mentioned, where $l = 2h$, we shall have

$$a = 2.152h;$$

$$c = .459h;$$

$$EC = 1.693h.$$

If it were required to find the form of the curve when a is 45^0, it might be obtained directly from the formula; which gives in this case

$$\frac{l}{h} = 1.1346.$$

51. PROB. II. *A chain of given length APB, fig.* 118, *is suspended from two given points A, B, not in the same horizontal line; to find its position.*

Let s represent the whole length of the chain, and x and y the ordinates of the point B, measured from A; and therefore given quantities. By equation (3), Art. 49, we have

$$a^2 + 2as \cdot \cos. a + s^2 = (a + x)^2 = a^2 + 2ax + x^2;$$

$$\therefore a = \frac{s^2 - x^2}{2(x - s\cos. a)}.$$

Hence $\sqrt{(a^2 + 2as \cdot \cos. a + s^2)} = a + x = \dfrac{s^2 - 2sx \cdot \cos. a + x^2}{2(x - s\cos. a)}$;

$$a\cos. a + s = \frac{2sx - s^2 \cdot \cos. a - x^2\cos. a}{2(x - s\cos. a)}.$$

But by (6) Art. 49,

$$y = a \cdot \sin. a \,\mathrm{l}\, \frac{\sqrt{(a^2 + 2as \cdot \cos. a + s^2)} + (a \cdot \cos. a + s)}{a(1 + \cos. a)},$$

$$= \frac{(s^2 - x^2) \cdot \sin. a}{2(x - s \cdot \cos. a)} \,\mathrm{l}\, \frac{(s^2 + 2sx + x^2) \cdot (1 - \cos. a)}{(s^2 - x^2) \cdot (1 + \cos. a)},$$

$$= \frac{(s^2 - x^2) \cdot \sin. a}{2(x - s \cdot \cos. a)} \,\mathrm{l}\, \frac{s + x}{s - x} \cdot \frac{1 - \cos. a}{1 + \cos. a};$$

whence a must be determined by approximation, as in the last problem. The approximation may be facilitated by the following artifice. Let $x = s \cdot \cos. \beta$; hence

$$\frac{y}{s} = \frac{\sin.^2\beta \cdot \sin. a}{2(\cos. \beta - \cos. a)} \cdot \mathrm{l}\, \frac{1 + \cos. \beta}{1 - \cos. \beta} \cdot \frac{1 - \cos. a}{1 + \cos. a}$$

whence

$$a(1-\cos.\alpha)\,\epsilon^{-\frac{y}{a\sin.\alpha}} = \sqrt{(s^2+2as\cos.\alpha+a^2)}-(s+a\cos.\alpha),$$

as appears by multiplying the equations. Hence, subtracting and dividing by 2,

$$s+a\cos.\alpha=\frac{a}{2}\left\{1+\cos.\alpha)\,\epsilon^{\frac{y}{a\sin.\alpha}}-(1-\cos.\alpha)\,\epsilon^{-\frac{y}{a\sin.\alpha}}\right\}\ldots(7).$$

Now

$$s+a\cos.\alpha=s'=c\frac{dx}{dy}=a\sin.\alpha\frac{dx}{dy},$$

Hence integrating both sides with regard to x, and dividing by $a\sin.\alpha$,

$$x+a=\frac{a}{2}\left\{(1+\cos.\alpha)\,\epsilon^{\frac{y}{a\sin.\alpha}}+(1-\cos.\alpha)\,\epsilon^{-\frac{y}{a\sin.\alpha}}\right\}\ldots\ldots(8)$$

And in nearly the same manner as before, we should find

$$y=a\cos.\alpha\,\mathrm{l}\,\frac{x+a\pm\sqrt{(x^2+2ax+a^2\cos.^2\alpha)}}{a(1\pm\cos.\alpha)}\ldots\ldots(9).$$

COR. It appears that $\dfrac{dy}{dx}=\dfrac{a\sin.\alpha}{a\cos.\alpha+s}$.

50. By means of the formulæ thus obtained we may solve the following Problems.

PROB. I. *A chain of given length $BCB'=2$l, fig. 118, hangs from two given points B, B' in the same horizontal line, of which the distance $BB'=2$h is given; to find its position.*

The middle point will here be the lowest, and the chain will form a symmetrical figure with respect to the axis CE;

$$CB=CB'=l,\ \ EB=EB'=h.$$

Let α be the angle which the curve at B makes with the vertical line; and by equation (3) Art. 47,

$$\frac{y}{c}=\mathrm{l}\left(\frac{s}{c}+\frac{\sqrt{(s^2+c^2)}}{c}\right)$$

Also by Cor. 2, Art. 46, tension at $B = \frac{s}{\cos. a} = BF$;

$$\therefore\ l = CB + BF = s + \frac{s}{\cos. a} = s\,\frac{1 + \cos. a}{\cos. a};$$

$$\therefore\ \frac{h}{l} = \frac{\sin. a}{1 + \cos. a}\,\mathrm{l}\,\frac{1 + \cos. a}{\sin. a};$$

$$\text{or, } \frac{h}{l} = \tan. \frac{a}{2} . \mathrm{l}\ \text{co-tan.}. \frac{a}{2} = -\tan. \frac{a}{2}\,\mathrm{l}\,\tan. \frac{a}{2};$$

whence $\tan. \frac{a}{2}$ must be found. And a being known, we know

$$s = l . \frac{\cos. a}{1 + \cos. a}: \text{ and } h = s \tan. a\,\mathrm{l}\,\frac{1 + \cos. a}{\sin. a};$$

and hence the curve is known.

53. Prob. IV. *In the last case, to find when the equilibrium is possible.*

In the equation $\frac{h}{l} = -\tan. \frac{a}{2}\,\mathrm{l}\,\tan. \frac{a}{2}$, making $\frac{l}{h} = u$, and $\tan. \frac{a}{2} = t$, we have $u = -\frac{1}{t\,\mathrm{l}\,t}$. And the relative changes of magnitude of t and u will be seen most easily by constructing a curve of which these shall be the abscissa and ordinate. Let bm, fig. 119, be always taken $= \tan. \frac{a}{2} = t$, and $mp = u$, and let us consider the locus of p.

The value of a will be between 0 and $\frac{1}{2}\pi$, and hence the value of t will be between 0 and 1; and hence $\mathrm{l}\,t$ will be negative, and u will always be positive

When $t = 0$, $t\,\mathrm{l}\,t = 0$, and u is infinite.

As t increases u decreases; we have

$$\frac{du}{dt} = \frac{1 + \mathrm{l}t}{(t\,\mathrm{l}\,t)^2};$$

which is negative so long as $-\mathrm{l}t > 1$.

When $1 + 1t = 0$, or $t = \frac{1}{e}$, $\frac{du}{dt} = 0$, and u is a minimum; at this point $u = \frac{l}{e}$, or $l = eh$. Afterwards u increases continually till $t = 1$, when u is infinite.

Hence the curve is of the form pqp', with asymptotes at b and c, bc being $= 1$. If $bn = .368$, &c. nq will be the minimum ordinate $= 2.718$, &c.

For every value, as bo, of u or $\frac{l}{h}$, there are two values of t, which may be found by drawing opp' parallel to bc. Hence there are *two* positions of equilibrium*, for given values of h and l. If we make bA perpendicular and equal to bc, and join Am, Am', the angles α for these two positions will be the doubles of bAm, bAm', respectively.

Thus, if $l = 10h$ the values of α are $3^0\,12'$, and $83^0\,36'$.

The least value of u or $\frac{l}{h}$ for which the equilibrium is possible, is when $u = e$, or $l = he$, which gives the minimum ordinate nq. In this case we have

$$t = \frac{1}{e}; \ \therefore \ \text{co-tan.}\frac{\alpha}{2} = \frac{1}{t} = 2.718281824, \text{\&c.};$$

$$\therefore \frac{\alpha}{2} = 20^0\,12', \text{ and } \alpha = 40^0\,24'.$$

If $a = BF$, fig. 118,

$$a = \frac{s}{\cos.\alpha} = \frac{l}{1+\cos.\alpha} = \frac{l}{2\cos.^2\frac{\alpha}{2}} = \frac{l}{2}\left(1 + \frac{1}{e^2}\right);$$

* In the case when the equilibrium is possible, the higher position is one of *stable*, the lower one of *unstable* equilibrium. If the chain be placed with its vertex above the higher of the two positions of equilibrium, it will descend towards that: if it be placed any where between the two positions, it will ascend towards the upper. If it be placed below the lower it will descend and never come to another position of equilibrium.

$$s = l - 2a = \frac{l}{2}\left(1 - \frac{1}{e^2}\right);$$

and by Art. 49, if k be the depth of the vertex below the horizontal line,

$$k = a - a \,.\, \sin. \alpha = \frac{l}{2} \,.\, \frac{1 - \sin. \alpha}{\cos.^2 \frac{\alpha}{2}} = \frac{l}{2}\left(\sec.^2 \frac{\alpha}{2} - 2 \tan. \frac{\alpha}{2}\right);$$

$$= \frac{l}{2} \,.\, \left(1 + \frac{1}{e^2} - \frac{2}{e}\right) = \frac{l}{2}\left(1 - \frac{1}{e}\right)^2,$$

$$\frac{s}{k} = \frac{e+1}{e-1}.$$

If the chain be so short, compared with the distance, that $\frac{l}{h}$ is less than e, it cannot be supported: the middle part will descend and draw up the ends.

54. Prob. V. *To find the center of gravity of the catenary AP*, fig. 118.

For this purpose we must find

$$\int_x x\frac{ds}{dx}, \quad \int_y y\frac{ds}{dy}.$$

By Cor. Art. 49,

$$\frac{dy}{dx} = \frac{c}{s'} = \frac{a \sin. \alpha}{a \cos. \alpha + s};$$

$$\therefore \; a \cos. \alpha + s = a \sin. \alpha \,.\, \frac{dx}{dy};$$

$$\therefore \; s = a\left(\sin. \alpha \,.\, \frac{dx}{dy} - \cos. \alpha\right),$$

$$\int_y s = a\,(x \sin. \alpha - y \cos. \alpha);$$

$$\therefore \; \int_y y\frac{ds}{dy} = ys - \int_y s = ys - a\,(x \sin. \alpha - y \cos. \alpha).$$

ain, since by (2) and (5), Art. 49,

$$\frac{dx}{ds'} = \frac{s'}{\sqrt{(c^2 + s'^2)}}, \quad \frac{dy}{ds'} = \frac{c}{\sqrt{(c^2 + s'^2)}};$$

$$\sqrt{(c^2 + s'^2)} = s' \frac{dx}{ds'} + c \frac{dy}{ds'};$$

and substituting $\frac{dx}{ds}$, $\frac{dy}{ds}$ for $\frac{dx}{ds'}$, $\frac{dy}{ds'}$;

ting for c and s' their values,

$$a^2 + 2as \cos. a + s^2) = (a \cos. a + s) \frac{dx}{ds} + a \sin. a . \frac{dy}{ds}.$$

ence by (3), Art. 49,

$$\textstyle\int_x x \frac{ds}{dx} = \int \{\sqrt{(a^2 + 2as \cos. a + s^2)} - a\} \frac{ds}{dx}$$

$$= \textstyle\int_s \left\{(a \cos. a + s) \frac{dx}{ds} + a \sin. a . \frac{dy}{ds} - a \frac{ds}{dx}\right\}.$$

ld $\int_x x \frac{ds}{dx}$ to both sides, and integrate, observing that

$$\textstyle\int_x \left(s + x \frac{ds}{dx}\right) = xs;$$

$$\therefore 2 \textstyle\int_x x \frac{ds}{dx} = ax \cos. a + xs + ay \sin. a - as.$$

ence by the formulæ, Art. 34,

$$h = \frac{\int_x x \frac{ds}{dx}}{s} = \frac{a(x \cos. a + y \sin. a)}{2s} - \frac{a - x}{2},$$

$$k = \frac{\int_y y \frac{ds}{dy}}{s} = y - \frac{a(x \sin. a - y \cos. a)}{s}.$$

COR. 1. It may be observed, that if we draw PO perpendicular on the tangent at P,

$$AO = x \cos. \alpha + y \sin. \alpha,$$

$$PO = x \sin. \alpha - y \cos. \alpha.$$

COR. 2. If we suppose A to be the lowest point, we have α a right angle. Hence

$$a + h = \frac{ay}{2s} + \frac{a + x}{2}.$$

which agrees with Art. 34, Ex. 23.

COR. 3. Let the tangents at A and P meet in T, and let TU be vertical; $PU = u$; $\therefore NU = y - u$, and if $PTU = \theta$, we shall have

$$u \text{ co-tan. } \theta + (y - u) \text{ co-tan. } \alpha = x.$$

$$\text{But co-tan. } \theta = \frac{dx}{dy} = \frac{a \cos. \alpha + s}{a \sin. \alpha}$$

$$= \text{co-tan. } \alpha + \frac{s}{a \sin. \alpha};$$

$$\therefore \frac{us}{a \sin. \alpha} + \frac{y \cos. \alpha}{\sin. \alpha} = x,$$

$$u = \frac{a (x \sin. \alpha - y \cos. \alpha)}{s};$$

$\therefore k + u = y$, and the center of gravity is in the vertical line TU which passes through T.

2. *The Catenary when the force acts in parallel lines and the Chain is not uniform.*

55. We may consider the thickness of the chain or cord to be variable, supposing it still to be so small throughout that we may consider the flexible body as a physical line. Or we may conceive the catenary to be a surface of unequal

breadth, resembling a ribbon, its breadth being parallel to the horizon; so that it may be a portion of a cylindrical surface, the curve of the cylinder being the catenary. We may also suppose the density to be variable. Or we may conceive the force which acts upon the chain, and gives weight to it, to be different in different parts.

Upon any of these suppositions the weight of equal portions of the curve taken in different parts of it will be different. Let $\frac{ds}{dy}$ be the differential coefficient of the curve with respect to y, and let $w\frac{ds}{dy}$ be the differential coefficient of the weight; w being the quantity (thickness, breadth, density or force) to which the weight of a given element of length is proportional.

Hence $\int_y w\frac{ds}{dy}$ taken between proper limits is the weight of any portion of the chain or string.

Prop. *To find the curve when the law of the thickness is given, and conversely.*

In fig. 117, let C be the lowest point, and let ma be the tension there. Then x and y being CM and MP as before, we shall have, as in Art. 46,

$$\frac{dx}{dy} = \frac{\int_y w\frac{ds}{dy}}{ma} \ldots\ldots\ldots\ldots(1).$$

If we differentiate this with respect to y, we have

$$\frac{d^2x}{dy^2} = \frac{w}{ma}\cdot\frac{ds}{dy} \ldots\ldots\ldots\ldots(2).$$

And w being known in terms of the other variable quantities, we shall, by integrating, have the equation to the curve.

$$\text{Also } w = ma\,.\frac{d^2x}{dy^2}\cdot\frac{dy}{ds} \ldots\ldots\ldots(3).$$

whence, if the curve be known, we may, by differentiating, find w.

56. PROB. VI. *A flexible string, whose thickness at every point is inversely as the square root of the length measured from the lowest point, is acted upon by gravity; to find its form.*

Let m be the thickness at a length c from the lowest point; hence, at the end of a length s,

$$w = m\frac{\sqrt{c}}{\sqrt{s}}; \quad \int_y w\frac{ds}{dy} = m\int_y \frac{\sqrt{c}}{\sqrt{s}}\cdot\frac{ds}{dy} = 2m\sqrt{(cs)};$$

$$\therefore \text{ by (1) } \frac{dx}{dy} = \frac{2\sqrt{(cs)}}{a}; \quad \left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = \frac{a^2+4cs}{a^2};$$

$$\frac{1}{a}\cdot\frac{dy}{ds} = \frac{1}{\sqrt{(a^2+4cs)}}; \text{ whence } \frac{y}{a} = \frac{\sqrt{(a^2+4cs)}}{2c} - \frac{a}{2c};$$

$$\left(\frac{y}{a}+\frac{a}{2c}\right)^2 = \frac{a^2}{4c^2} + \frac{s}{c}; \quad \frac{s}{c} = \frac{y^2}{a^2} + \frac{y}{c}.$$

$$\text{Hence } \frac{1}{c}\cdot\frac{ds}{dy} = \left(\frac{2y}{a^2}+\frac{1}{c}\right); \quad \frac{1}{c^2}\left\{1+\left(\frac{dx}{dy}\right)^2\right\} = \left(\frac{2y}{a^2}+\frac{1}{c}\right)^2;$$

$$\frac{1}{c^2}\left(\frac{dx}{dy}\right)^2 = \left(\frac{4y^2}{a^4}+\frac{4y}{a^2c}\right); \quad \frac{dx}{dy} = \frac{2c}{a^2}\sqrt{\left(y^2+\frac{1}{c}y\right)};$$

whence y is easily found by integrating; and hence the curve is known.

57. PROB. VII. *A flexible string is acted on by a force which is, at every point, as the height above the lowest point: to find its form.*

Let the origin be as before: and at the height c above the lowest point let the force be m; hence, at the height x, since *cæteris paribus* the weight of any portion will be as the force,

$$w = \frac{mx}{c}; \quad \therefore \text{ by (1) } \frac{dx}{dy} = \frac{\int_y x\frac{ds}{dy}}{ca}, \quad \frac{d^2x}{dy^2} = \frac{x}{ca}\cdot\frac{ds}{dy};$$

$$\frac{\dfrac{d^2x}{dy^2}}{\sqrt{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}}}=\frac{x}{ca}; \quad \frac{\dfrac{dx}{dy}\cdot\dfrac{d^2x}{dy^2}}{\sqrt{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}}}=\frac{x}{ca}\cdot\frac{dx}{dy};$$

Integrating with respect to y,

$$\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}=\frac{x^2}{2ca}+1; \quad \frac{dx}{dy}=\sqrt{\left(\frac{x^4}{4c^2a^2}+\frac{x^2}{ca}\right)};$$

$$\frac{dy}{dx}=\frac{2ac}{x\sqrt{(x^2+4ac)}}; \quad y=\frac{\sqrt{(ac)}}{2}\,\mathrm{l}\,\frac{\sqrt{(x^2+4ac)}-2\sqrt{(ac)}}{\sqrt{(x^2+4ac)}+2\sqrt{(ac)}}+\text{const.}$$

When $x=0$, y is infinite and negative; when x is infinite, y is equal to the constant. Hence the curve has a vertical and a horizontal asymptote, and never meets the horizontal line in which the force is $=0$.

58. PROB. VIII. *To find the law of thickness of a string that it may hang in the form of a semi-circle.*

Placing the origin at the lowest point, as before, we must have, calling the radius of the circle c,

$$y=\sqrt{(2cx-x^2)}; \quad \therefore \frac{dx}{dy}=\frac{\sqrt{(2cx-x^2)}}{c-x}, \quad \frac{ds}{dy}=\frac{c}{c-x};$$

also, $$\frac{d^2x}{dy^2}=\frac{d\left(\dfrac{dx}{dy}\right)}{dy}=\frac{d\left(\dfrac{dx}{dy}\right)}{dx}\cdot\frac{dx}{dy}=\frac{c^2}{(c-x)^2\sqrt{(2cx-x^2)}}\,\frac{dx}{dy}$$

$$=\frac{c^2}{(c-x)^3}; \quad \text{also } \frac{dy}{ds}=\frac{c-x}{c};$$

whence by (3), $$w=ma\cdot\frac{d^2x}{dy^2}\cdot\frac{dy}{ds}=\frac{mac}{(c-x)^2};$$

hence the thickness must vary inversely as the square of the depth below the horizontal diameter.

The tension will be found, as before, by the equation,

$$\text{tension}=ma\cdot\frac{ds}{dy}=\frac{mac}{c-x}.$$

Hence, at the extremities of the horizontal diameter it is infinite.

If, instead of supposing the thickness of the string to vary, we suppose to be hung to each point of it vertical strings of uniform thickness whose lengths are proportional to

$$\frac{mac}{(c-x)^2},$$

the curve which it will form will be the same. And this is also applicable to all the cases of this section.

59. **Prob. IX.** *To find the law of thickness of a string that it may hang in the form of a parabola with its axis vertical.*

The origin is at the lowest point as before:

$$\text{By (3)}, \quad w = ma \cdot \frac{d^2x}{dy^2} \cdot \frac{ds}{dy},$$

$$\text{and } y^2 = 4cx; \ \therefore \ \frac{dy}{dx} = \frac{\sqrt{c}}{\sqrt{x}}; \ \frac{ds}{dy} = \frac{\sqrt{(x+c)}}{\sqrt{c}};$$

$$\frac{d^2x}{dy^2} = \frac{1}{2\sqrt{(cx)}} \cdot \frac{dx}{dy} = \frac{1}{2c};$$

$$\therefore \ w = \frac{ma}{2\sqrt{(cx+c^2)}}.$$

When $x = 0$, $w = \dfrac{ma}{2c}$; and if m be the thickness at the lowest point, $a = 2c$,

$$w = \frac{m\sqrt{c}}{\sqrt{(x+c)}}.$$

So long as x is small, this is nearly constant. Hence, conversely, if the thickness be constant, the catenary, within a small distance of the vertex, nearly coincides with a parabola. This is a conclusion to which Galileo was led by experiment.

3. *The Catenary when the Chain is acted upon by a central attractive or repulsive force*.*

60. Prop. *To find the equation to the catenary when the force tends to a center.*

Let S, fig. 120, be the center of attractive force, and at any distance $SP = r$, let the force be $= f$, f being a function of r. Let $AP = s$ be the chain or cord, and at the point P let the mass of a small particle δs be $\mu \delta s$, μ depending upon the thickness, density, &c.

Let A be the point at which the curve is perpendicular to SA. Make SA a line of abscissas, and let MP be an ordinate perpendicular to it: Py a tangent at P, and Sy perpendicular on it.

Put $Sy = p$; tension at $A = a$, tension at $P = t$; angle $ASP = \theta$, $ATP = \phi$. The weight of a particle ds at P will be $f\mu$ in the direction PS, (see Note:) and if we resolve this force in the directions parallel and perpendicular to AS, the components will be $f\mu \cos. \theta$ and $f\mu \sin. \theta$: and hence the whole effects of the weight in those directions will be $\int_s f\mu \cos. \theta$ and $\int_s f\mu \sin. \theta$. The other forces which act on the cord AP, are the tension at $A = a$; and the tension at $P = t$, which may be resolved into the parts $t . \cos. \phi$ and $t . \sin \phi$, in the rectangular directions. As before, the forces which keep AP at rest must be subject to the conditions of Art. 22. Hence,

$$\left.\begin{aligned} \int_s f\mu \cos. \theta &= t \cos. \phi \\ \int_s f\mu \sin. \theta &= t \sin. \phi - a \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1).$$

Differentiating with respect to s,

$$f\mu \cos. \theta = \frac{dt}{ds} . \cos. \phi - t \sin. \phi \frac{d\phi}{ds},$$

* The force spoken of here and in the last Article is the attractive force which produces weight or pressure in the bodies on which it acts. If other things remain the same, such attractive forces are as the weight which they produce in a given particle of matter.

$$f\mu \sin.\ \theta = \frac{dt}{ds} \cdot \sin.\ \phi + t \cos.\ \phi \frac{d\phi}{ds}.$$

Multiply the first by cos. ϕ, and the second by sin. ϕ, and add; and we have

$$f\mu \cos.\ (\phi - \theta) = \frac{dt}{ds} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying the first by sin. ϕ, and the second by cos. ϕ, and subtract; and we have

$$f\mu \sin.\ (\phi - \theta) = -t\frac{d\phi}{ds} \ldots\ldots\ldots\ldots\ldots (3).$$

But $\phi - \theta = ATP - TSP = SPy$. Also if we take PQ a small arc, and draw Qn perpendicular on SP, $\frac{Pn}{PQ}, \frac{Qn}{PQ}$, are ultimately as $\frac{dr}{ds}, r\frac{d\theta}{ds}$. Hence

$$\cos.\ (\phi - \theta) = \frac{dr}{ds}, \quad \sin.\ (\phi - \theta) = r\frac{d\theta}{ds},$$

and (2) and (3) become

$$f\mu \frac{dr}{ds} = \frac{dt}{ds}, \text{ or } f\mu = \frac{dt}{dr} \ldots\ldots\ldots\ldots\ldots (4)$$

$$f\mu r\frac{d\theta}{ds} = -t\frac{d\phi}{ds}. \text{ or } f\mu r = -t\frac{d\phi}{d\theta} \ldots. (5).$$

$$\text{Now } r\frac{d\theta}{dr} = \frac{Qn}{Pn} = \frac{p}{\sqrt{(r^2 - p^2)}};$$

$$\therefore \frac{d\theta}{dr} = \frac{p}{r\sqrt{(r^2 - p^2)}}.$$

$$\text{Also } \phi - \theta = \text{arc.}\left(\sin. = \frac{p}{r}\right);$$

$$\therefore \frac{d\phi}{dr} - \frac{d\theta}{dr} = \frac{r\frac{dp}{dr} - p}{r\sqrt{(r^2 - p^2)}};$$

$$\therefore \text{ adding, } \frac{d\phi}{dr} = \frac{\frac{dp}{dr}}{\sqrt{(r^2 - p^2)}}.$$

Also (5) is equivalent to

$$f\mu r \frac{d\theta}{dr} = -t \frac{d\theta}{dr}.$$

Putting the values of $\frac{d\theta}{dr}$ and $\frac{d\phi}{dr}$ in this, it becomes, reducing,

$$f\mu p = -t \frac{dp}{dr} \text{..........................(6)}.$$

Dividing (6) by (4) we have

$$p = -\frac{t \frac{dp}{dr}}{\frac{dt}{dr}};$$

$$\therefore p \frac{dt}{dr} + t \frac{dp}{dr} = 0;$$

$$pt = C^{*} \text{..............................(7)},$$

C being a constant quantity, to be determined by the conditions of the question.

Also we have, by (4),

$$t = \int_r f\mu.$$

$$\text{Hence } p = \frac{C}{\int_r f\mu}.$$

When f is known in terms of r, this equation gives the curve AP, by an equation between the distance SP and the

* The property, that the perpendicular is inversely as the tension, appears also from this, that AP is acted on by the tensions at A and P, and also by central forces all tending to S. Hence the result of these latter forces will also tend to S; and hence we may suppose AP retained by a lever passing through S as a fulcrum, and the two forces at A and at P will be inversely as the perpendiculars on their directions; therefore tension at $P . Sy =$ tension at $A . SA =$ a constant quantity.

perpendicular Sy upon the tangent. And from this equation we may determine the relation between r and θ; and between x and y: unless this is rendered impossible by the difficulty of integrating.

The tension $t = \int_r f\mu$; if the thickness and density be constant, we may make $\mu = 1$, and $t = \int_r f$; hence the tension depends only on the distance r, and is not affected by the form of the curve. If we suppose the end Pp to hang freely over the point P, and thus to produce the equilibrium, its weight must be $\int_r f$; which is also the weight of a string extending from a point p, at a given distance from S, up to P. Hence at every point P the string Pp will hang to the same distance Sp from S; and the ends of all the strings will be in a circle with center S; in which circle also is the point a, Aa being the length whose weight is requisite to produce the tension at A.

61. PROB. X. *The force varying inversely as the square of the distance from S, it is required to find the form of the catenary.*

Let SA, fig. 120, $= c$, and the force at $A = k$; hence

$$f = \frac{kc^2}{r^2};$$

$$t = \int_r f = \int_r \frac{kc^2}{r^2} = \text{constant} - \frac{kc^2}{r} = a + kc - \frac{kc^2}{r};$$

for when $r = c$, $t = a$.

$$\text{Hence } p = \frac{C}{a + kc - \dfrac{kc^2}{r}} = \frac{ac}{a + kc - \dfrac{kc^2}{r}},$$

for when $r = c$, $p = c$,

$$= \frac{acr}{(a + kc)\, r - kc^2}.$$

Let $a = nkc$; kc being the weight of a length of string AS, acted on by a constant force equal to that at A; hence

$$p = \frac{ncr}{(n+1)\, r - c}.$$

To determine the nature of the curve, we have

$$\frac{d\theta}{dr}=\frac{1}{r\sqrt{\left(\frac{r^2}{p^2}-1\right)}}=\frac{nc}{r\sqrt{\{(n+1)^2r^2-2(n+1)cr+c^2-n^2c^2\}}};$$

which will give different forms as n is greater than, equal to, or less than unity.

(1.) Let $n>1$; therefore

$$\frac{d\theta}{dr}=\frac{nc}{r^2\sqrt{\left((n+1)^2-2(n+1)\cdot\frac{c}{r}-(n^2-1)\frac{c^2}{r^2}\right)}};$$

which may be integrated by making $\frac{n-1}{n}\cdot\frac{c}{r}+\frac{1}{n}=u$, and gives

$$\theta=\frac{n}{\sqrt{(n^2-1)}}\cdot\text{arc.}\left(\cos.=\frac{(n-1)c+r}{nr}\right);$$

θ being measured from the line SA.

If we make r infinite, we have

$$\theta=\frac{n}{\sqrt{(n^2-1)}}\cdot\text{arc.}\left(\cos.=\frac{1}{n}\right);$$

which gives the position of the asymptotes of the curve.

The angle which the asymptotes make with SA, is greater, as n, and consequently the tension at A, is greater. When n is infinite, it is a right angle, and the curve becomes a straight line perpendicular to SA. As n diminishes to unity, θ diminishes to the value which it has in the next case.

(2.) Let $n=1$: hence

$$\frac{d\theta}{dr}=\frac{c}{2r\sqrt{(r^2-cr)}}=\frac{c}{2r^2\sqrt{\left(1-\frac{c}{r}\right)}};$$

which gives, by integrating,

$$\theta = \sqrt{\left(1 - \frac{c}{r}\right)} + \text{const.} = \sqrt{\left(1 - \frac{c}{r}\right)};$$

because $\theta = 0$ when $r = c$.

When r is infinite $\theta = 1$. Hence the angle which the asymptotes make with SA is that whose arc is equal to the radius; or, if RO be the asymptote,

$$ARO = 57^0\ 14'\ 44''\ 48'''.$$

In every case we may find the position of the asymptote by making r infinite in the value of p; which will give

$$S\aleph = \frac{nc}{n+1}.$$

(3.) Let $n < 1$: hence

$$\frac{d\theta}{dr} = \frac{nc}{r^2\sqrt{\left((1+n)^2 - 2(1+n)\frac{c}{r} + (1-n^2)\frac{c^2}{r^2}\right)}};$$

which may be integrated by making $1 - (1-n)\frac{c}{r} = u$; and gives $\theta =$

$$\frac{n}{\sqrt{(1-n^2)}}\,\mathrm{l}\,\frac{r-(1-n)\,c+\sqrt{\{(1-n^2)\,r^2-2cr+(1-n)^2c^2\}}}{nr};$$

the integral being corrected so as to vanish when $r = c$.

When r is infinite, $\theta = \dfrac{n}{\sqrt{(1-n^2)}}\,.\,\mathrm{l}\,\dfrac{1+\sqrt{(1-n^2)}}{n}$, which gives the position of the asymptote. When $n = 1$, $\theta = 1$, as may easily be shewn, agreeably to the last case. As n diminishes, the angle which the asymptotes make with SA diminishes, and when n becomes 0 this angle vanishes.

The tension at A is equal to the weight of a string whose length is nc, acted upon by a constant force equal to that at A. But if $Sa = b$, the weight of the portion Aa, acted

on by the variable force (which weight expresses the tension at A) will be

$$= \int_r \frac{kc^2}{r^2}, \text{ the integral taken from } r = b, \text{ to } r = c$$

$$= \frac{kc^2}{b} - \frac{kc^2}{c} = nkc, \text{ by supposition;}$$

$$\therefore b = \frac{c}{1+n}.$$

Hence if a circle were described with a radius $Aa = b$, the string hanging down from any point of the curve, must, in order to produce the tension at that point, reach to the circumference of this circle.

62. Prob. XI. *Let the force vary as the* m[th] *power of the distance from S: to find the curve.*

Retaining the notation of the last Problem, we have

$$\text{force at } P = \frac{kr^m}{c^m}; \ \therefore t = \int_r \frac{kr^m}{c^m} = \frac{kr^{m+1}}{(m+1)c^m} + \text{const.}$$

$$= a - \frac{kc}{m+1} + \frac{kr^{m+1}}{(m+1)c^m};$$

a being the tension at A. Let $a = \dfrac{nkc}{m+1}$; therefore

$$t = \frac{kc}{m+1}\left(n - 1 + \frac{r^{m+1}}{c^{m+1}}\right).$$

Hence

$$p = \frac{nc}{n - 1 + \frac{r^{m+1}}{c^{m+1}}} = \frac{nc^{m+2}}{(n-1)\,c^{m+1} + r^{m+1}}:$$

and

$$\frac{d\theta}{dr} = \frac{1}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}} = \frac{nc^{m+2}}{r\sqrt{[\{r^{m+2} + (n-1)c^{m+1}r\}^2 - n^2c^{2m+4}]}},$$

which cannot be integrated generally except $n = 1$.

In the case of $n = 1$,

$$\frac{d\theta}{dr} = \frac{c^{m+2}}{r\sqrt{\{r^{2m+4} - c^{2m+4}\}}};$$

which may be integrated by making $c^{m+2}u = r^{m+2}$: this substitution gives

$$\frac{d\theta}{dr} = (m+2)u\sqrt{(u^2 - 1)};$$

$$\therefore \theta = \frac{1}{m+2} . \text{arc} (\text{sec.} = u)$$

$$= \frac{1}{(m+2} \text{arc} \left(\text{sec.} = \frac{r^{m+2}}{c^{m+2}}\right);$$

$$\frac{r^{m+2}}{c^{m+2}} = \text{sec.} (m+2)\theta; \quad r^{m+2} \cos. (m+2)\theta = c^{m+2}.$$

If we make r infinite, we have for the inclination of the asymptotes to SA,

$$\theta = \frac{\pi}{2m+4}.$$

63. We may find the equation between

$$SM = x \text{ and } MP = y.$$

For $$r = \sqrt{(x^2 + y^2)},$$

$$\cos. \theta = \frac{x}{\sqrt{(x^2 + y^2)}}, \quad \tan. \theta = \frac{y}{x}.$$

Hence

$$c^{m+2} = r^{m+2} \cos. (m+2)\,\theta$$

$$= r^{m+2}. \left(\cos.^{m+2}\theta - \frac{(m+2)(m+1)}{1.2} \cos.^{m}\theta . \sin^{2}. \theta\right.$$

$$\left. + \frac{(m+2)(m+1)m(m-1)}{1.2.3.4} \cos.^{m-2}\theta . \sin.^{4}\theta ......\right)$$

$$= r^{m+2} \cos.^{m+2}\theta . \left(1 - \frac{(m+2)(m+1)}{1.2} \tan.^{2}\theta\right.$$

$$\left. + \frac{(m+2)......(m-1)}{1.2.3.4} \tan.^{4}\theta - ......\right)$$

$$x^{m+2}\left(1 - \frac{(m+2)(m+1)}{1.2} . \frac{y^2}{x^2} + \frac{(m+2)...(m-1)}{1.2.3.4} . \frac{y^4}{x^4} - ...\right).$$

Cor. 1. If $m = 0$, or the force be constant,

$$c^2 = x^2\left(1 - \frac{y^2}{x^2}\right) = x^2 - y^2.$$

Hence the curve is the rectangular hyperbola. The symptotes make angles of 45^0 with SA.

Cor. 2. If $m = 1$, or the force be as the distance,

$$c^3 = x^3\left(1 - \frac{3y^3}{x^3}\right) = x^3 - 3y^3.$$

In this case the angle which the asymptotes make with SA is $34^0\,44'$.

64. Prop. *To find the catenary when the central force is repulsive.*

The process for finding the curve of equilibrium in this case will be nearly the same as before, with the excep-

tion of the signs of some of the quantities, and the results will be

$$-f\mu = \frac{dt}{dr};$$

$$f\mu p = t\frac{dp}{dr}:$$

$$\therefore \text{ dividing, } \quad \frac{\frac{dp}{dr}}{p} = \frac{\frac{dt}{dr}}{t}, \quad \therefore p = \frac{C}{t},$$

$$\text{and } t = \textstyle\int_r f\mu; \quad \therefore p = \frac{C}{-\int_r f\mu}.$$

65. PROB. XII. *S, fig.* 121, *is a center of repulsive force varying inversely as the square of the distance from S: to find the form of the curve AP, formed by a flexible string.*

Retaining the notation of Prob. X,

$$t = -\int_r \frac{kc^2}{r^2} = a - kc + \frac{kc^2}{r};$$

a being the tension at A: let $a = nkc$;

$$\therefore t = kc\left((n-1) + \frac{c}{r}\right).$$

Hence $p = \dfrac{ncr}{(n-1)r + c}$, supposing the curve at A perpendicular to SA;

$$\therefore \frac{d\theta}{dr} = -\frac{1}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}}$$

$$= -\frac{nc}{r\sqrt{\{(n-1)^2 r^2 + 2(n-1)cr + c^2 - n^2c^2\}}},$$

which may be integrated nearly as in Prob. X.

If we suppose the curve not to be perpendicular to SA, ; to make with it an angle a, we shall have at that point : $c \cdot \sin. a$;

$$\therefore p = \frac{ncr \cdot \sin. a}{(n-1)\, r + c}.$$

If $n = 1$, this becomes $p = r \cdot \sin. a$, and the curve is the 'arithmic spiral.

66. PROB. XIII. *Let the force be inversely as the* m[th] *wer of the distance: to find the curve.*

$$t = -\int_r \frac{kc^m}{r^m} = a - \frac{kc}{m-1} + \frac{kc^m}{(m-1) r^{m-1}};$$

$$= \frac{kc}{m-1} \cdot \left(n - + \frac{c^{m-1}}{r^{m-1}}\right);$$

$$\text{putting } a = \frac{nkc}{m-1}.$$

Take the case when $n = 1$, and we have

$$t = \frac{kc^{m-1}}{(m-1)\, r^{m-1}}; \text{ and let } p = c \cdot \sin. a, \text{ at } A;$$

$$\therefore p = \frac{C}{t} = \frac{r^{m-1} \cdot \sin. a}{c^{m-2}};$$

$$\frac{d\theta}{dr} = -\frac{1}{r\sqrt{\left(\dfrac{r^2}{p^2} - 1\right)}} = -\frac{r^{m-2} \sin. a}{r\sqrt{(c^{2m-4} - r^{2m-4} \sin.^2 a)}}$$

$$= -\frac{r^{m-3} \sin. a}{\sqrt{(c^{2m-4} - r^{2m-4} \sin.^2 a)}}.$$

To integrate, put $\dfrac{r^{m-2} \sin. a}{c^{m-2}} = u$;

$$\therefore \frac{d\theta}{du} = -\frac{1}{(m-2)\sqrt{(1-u^2)}},$$

$$\theta = \frac{1}{m-2}\text{ arc }(\cos. = u) = \frac{1}{m-1}\text{ arc }\left(\cos. = \frac{r^{m-2}\sin. a}{c^{m-2}}\right).$$

$$\text{Or, if } \frac{c^{m-2}}{\sin. a} = a^{m-2},\ (m-2)\,\theta \text{ arc }\left(\cos. = \frac{r^{m-2}}{a^{m-2}}\right).$$

Here a is the value of r at the point A.

To find the angle to $2ASO$ which comprehends the whole curve, make $r = 0$:

$$\therefore\ \theta = \frac{\pi}{2m-4}; \quad \therefore\ 2ASO = 2\theta = \frac{\pi}{m-2}.$$

We may find the equation between $SM = x$, and $MA = y$, as before,

For $r^{m-2} = a^{m-2}\cos.(m-2)\,\theta$

$$= a^{m-2}\left(\cos.^{m-2}\theta - \frac{(m-2)(m-3)}{1.2}\cos.^{m-4}\theta.\sin.^2\theta + \ldots\right);$$

$$\therefore\ r^{2m-4} = a^{m-}r^{2m-2}\cos.^{m-2}\theta\left(1 - \frac{(m-2)(m-3)}{1.2}\tan.^2\theta + \ldots\right);$$

$$\text{or } (x^2+y^2)^{m-2} = a^{m-2}x^{m-2}\left(1 - \frac{(m-2)(m-3)}{1.2}.\frac{y^2}{x^2} + \ldots\right).$$

COR. 1. If $m = 3$, $\theta = \text{arc}\left(\cos. = \dfrac{r}{a}\right)$; hence APS is a circle on the diameter AS.

COR. 2. If $m = 4$, $2\theta = \text{arc}\left(\cos. = \dfrac{r^2}{a^2}\right)$; hence APS is the lemniscata with its knot at S.

COR. 3. Hence if there be a centre of repulsive force which varies inversely as the cube of the distance, and if the two ends of a string be fastened at this center, it will form itself into a circle. If the force vary inversely as the fourth power, the curve will be a lemniscata, and so on.

The Catenary when the Chain is acted upon by any Forces.

67. PROP. *Let forces to act upon the flexible body AP,* 122, *in the same plane, accordingto any law whatever;* *required to find its form.*

Let the force at any point P be represented by f, and in the direction PF, which makes with the line of abas AM an angle ψ. The reasoning is exactly the same Art. 60. The effect of the force f at P is fds, and this, lved parallel and perpendicular to AM, gives $f\cos.\psi$, $f\sin.\psi$. Hence the whole effects on AP are $\int_s f\cos.\psi$, $\int_s f\sin.\psi$. The remaining forces are the tension at P, h is represented by t, and makes with AM an angle ϕ, the tension at A, which is represented by a, and is supd to be perpendicular to AM. Hence the conditions of 23, give

$$\int_s f\cos.\psi - t\cos.\phi = 0,$$

$$\int_s f\sin.\psi + t\sin.\phi = a.$$

Differentiating,

$$f\cos.\psi - \frac{dt}{ds}\cos.\phi + t\sin.\phi\,.\frac{d\phi}{ds} = 0.$$

$$f\sin.\psi + \frac{dt}{ds}\sin.\phi + t\cos.\phi\,.\frac{d\phi}{ds} = 0.$$

Multiply the first by $\cos.\phi$ and the second by $\sin.\phi$, and tract: also multiply the first by $\sin.\phi$ and the second by ϕ, and add: we shall thus get

$$f(\cos.\phi\,.\cos.\psi - \sin.\phi\,.\sin.\psi) - \frac{dt}{ds} = 0;$$

$$f(\sin.\phi\,.\cos.\psi + \cos.\phi\,.\sin.\psi) + t\frac{d\phi}{ds} = 0;$$

$$\text{or, } f\cos.(\phi+\psi) = \frac{dt}{ds},$$

$$f\sin.(\phi+\psi) = -t\frac{d\phi}{ds}.$$

The angle $\phi + \psi$ is FPT, the angle which the force makes with the tangent. This angle and the force f being expressed in terms of x and y and their differentials, t is known from the first equation: and this value of t, and $\frac{d\phi}{ds}$, being substituted in the second, we have the equation to the curve. For $\frac{d\phi}{ds}$, we have

$$\phi = \text{arc}\left(\tan. = \frac{dy}{dx}\right);$$

$$\therefore \frac{d\phi}{ds} = \frac{d\phi}{dx}\frac{dx}{ds} = \frac{\frac{d^2y}{dx}\cdot\frac{dx}{ds}}{1+\left(\frac{dy}{dx}\right)^2} = \frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3};$$

the differentiations being performed with respect to x.

68. Prop. *If the force be at every point perpendicular to the curve, to find the form.*

We shall have $\phi + \psi = \frac{1}{2}\pi$; hence cos. $(\phi + \psi) = 0$, sin. $(\phi + \psi) = 1$; and our equations become

$$0 = \frac{dt}{ds};$$

$$f = -t\frac{d\phi}{ds},$$

$$\text{or } t = a;$$

$$f = -a.\frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3} = \frac{a}{\rho}; \quad \therefore a = f\rho,$$

ρ being the radius of curvature.

Hence *when the force is perpendicular to the curve, the tension is constant;* and is at every point equal to the weight of a portion of the cord, whose length is the radius of curvature, acted on by the force at that point. If curvature be

supposed inversely proportional to the radius of curvature, *the curvature at every point will be as the force.*

69. Prob. XIV. *A flexible line AP, fig.* 122, *is acted upon at every point P by a force* f, *which is perpendicular to the line, and which is as the square of the sine of the angle EPV; to find the curve AP.*

The sine of $EPV = \sin.\phi = \dfrac{dy}{ds}$; hence the force

$$f = k\left(\frac{dy}{ds^2}\right)^2,$$

k being its value at A;

$$\therefore\ k.\left(\frac{dy}{ds^2}\right)^2 = -a\frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{ds}{dx}\right)^3};$$

$$\therefore\ k\frac{ds}{dx} = -\frac{a\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^2};$$

$$\therefore\ ks + \text{const.} = a : \frac{1}{\dfrac{dy}{dx}} = a\frac{dx}{dy};\quad \text{also at } A,\ s = 0,\ \frac{dx}{dy} = 0;$$

$$\therefore\ \text{const.} = 0;$$

$$\therefore\ \frac{ks}{a} = \frac{dx}{dy};$$

which coincides with the equation to the common catenary when the origin is placed at the lowest point and x taken vertical. Hence this is the same curve as when the force is parallel and constant*.

* Soon after the time (1691) when the Problem of the figure of a chain acted upon by gravity was proposed and solved by the Bernoullis and Leibnitz, the attention of these geometers was directed to other curves which flexible bodies may assume under various circumstances. In particular the action of a fluid, whether by elasticity, weight, or impact, was considered; and as this action must be perpendicular to the

70. PROB. XV. *AP is acted upon by forces which are every where perpendicular to the curve, and which are, at every point P, proportional to the distance PE of P from a given line BE; to find the curve.*

Let BE be perpendicular to AB, $AB = c$; $PE = x$; $k =$ the force at A;

$$\therefore \frac{kx}{c} = -a \frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{ds}{dx}\right)^3};$$

which coincides with the equation to the elastic curve, as will be seen in the next Chapter, where that curve is considered.

We might now proceed to consider more complicated cases, as for instance when the flexible string rests upon any

---

surface on which it acts, this case comes under Art. 68. of the text. One of their problems was, To find the figure of a rectangular *sail*, with two opposite sides fixed, inflated by the wind: and as the figure of a chain or cord had been called the *Catenaria* or *Funicularia*, this was called the *Velaria*. The weight of the sail itself being neglected, the problem may be solved on either of the following hypotheses:

1st, That the air which immediately presses the sail is, relatively to the sail, at rest; and of course kept in its place by the pressure produced by the wind behind. On this supposition it is the elasticity of the air which acts upon the curve; and since this force is the same at every point, the radius of curvature will be constant, and the curve will be a circular arc; consequently the surface will be a portion of a common cylinder.

2nd, That the air acts by impact, and produces no effect by pressure after the first impulse. This may be nearly the case when a single thread is stretched by a current of fluid, which can after the impact escape past it. In this case the force is as the square of the sine of the angle of impact, as appears from hydrodynamical principles. Hence this is the case of Prob. XIV, of the text, in which as is shewn, the curve is the common *Catenaria*.

It appears to have been supposed that the actual curve of the sail would be something compounded of both these forms.

Another problem of the same kind was, To find the form of a rectangle of cloth, &c. which having two opposite sides supported parallel to the horizon, is pressed by the weight of a fluid which is contained in it, and of course supposed to be prevented from running out at the ends. The curve of this problem was called the *Lintearia*; if BC, fig. 122, be the surface of the fluid, the pressure on any point P will be as the depth EP; hence the curve is the one found in Prob. XV; which, as is mentioned in the text, is the same with the *Elastica*.

curve surface or surfaces. We might also investigate the conditions of equilibrium of a flexible *surface* acted upon by gravity or by any forces. The mechanical principles of such problems would not present much difficulty after what has preceded, but the analytical results to which they would lead would in most cases be too complicated for an elementary work like the present

On Suspension Bridges.

71. The curve formed by the chains or cords by which the road-way of a suspension bridge is supported will be a catenary if the weight supported by each part of the chain (namely the suspension rods, road-way, &c.) be proportional to the length of that part of the chain. We shall first consider the case of a suspension bridge on this supposition.

The tenacity of iron is such that a rod of 1 inch section will support the weight 14800 feet of the same rod; and the same is true for any other section: hence 14800 feet is the *modulus of tenacity* of this substance, and in like manner the tenacity of any other substance may be expressed by the length of the rod of the material of uniform thickness which the tenacity will support; and this length is the modulus of tenacity for that material.

The tension of the catenary at its vertex is represented by a length c, (Art. 46.) of the chain or cord, such that the weight of this length is equal to the force of the tension. And if the chain be loaded with additional weights distributed uniformly along its length, the tension at the vertex and at every other point will be increased in a constant proportion, that is, in the proportion of the augmented weight of any part.

72. PROP. *Given the width of a suspension bridge, to find its dimensions, so that the chains shall nowhere be loaded with more than a given fraction* (n) *of the weight they are able to sustain.*

Let c be the tension at the vertex, supposing the chain to support only its own weight; x the vertical abscissa from the vertex, y the horizontal ordinate: then the tension at any point is $x + c$. But if the weight of the chain be increased at every point (by the rods, road, &c.) in a constant ratio $1 + m : 1$, the tension at the vertex will be $(1 + m)c$, and at any other point $(1 + m)(x + c)$. Let l be the modulus of tenacity of the substance; the rods, road, &c. are supposed to add nothing to the tenacity of the chains. Hence l is the ultimate limit which the tension can attain, and nl is the limit which is prescribed in the proposition. Therefore

$(1 + m)(x + c)$ must be less than nl.

Now y is given, being the half width of the bridge. And (Art. 48.)

$$x + c = \frac{c}{2}\left(\epsilon^{\frac{y}{c}} + \epsilon^{-\frac{y}{c}}\right).$$

Let N be the number of which $\frac{y}{c}$ is the natural logarithm,

$$\text{and } \epsilon^{\frac{y}{c}} = N \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.)}$$

Hence

$$x = \frac{c}{2}\left(N + \frac{1}{N}\right) - c = c\,\frac{(N-1)^2}{2N} \ldots\ldots (2).$$

A simple mode of finding x from this is the following.

Assume $y = 100$, and c equal to various values, at convenient intervals from 0 to 1000, or further if necessary.

Construct by formulæ (1) and (2) a Table of the values of x corresponding to each value of c; and of the corresponding values of the tension t at the extremity of the arc, which is always $x + c$.

The strength nl being expressed in the same scale in which y is 100, we shall find in the Table the greatest value of $(1 + m)(x + c)$ which is less than nl; and this gives the greatest allowable value of x, the depth of the vertex of the curve below the points of suspension.

If α be the angle which the curve at the points of suspension makes with the vertical; by Art. 48,

$$\text{cotan. } \alpha = \frac{dx}{dy} = \tfrac{1}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{-\frac{y}{c}}\right) = \frac{N^2 - 1}{2N}.$$

And hence α may be found, and inserted in the same Table.

73. A Table such as we have described is inserted in the *Phil. Trans.* for 1826, p. 213, by Mr Davies Gilbert. The following is an extract from it. In addition to the quantities mentioned above, s is the length of the chain from the lowest point to the extremity.

$y = 100$

c	N	x	s	t	α
1000	1.11	5.00	100.16	1005.00	84° 16′
900	1.12	5.56	100.21	905.56	83 38
800	1.13	6.26	100.26	806.26	82 51
700	1.15	7.15	100.34	707.15	81 50
600	1.18	8.35	100.46	608.35	80 30
500	1.22	10.03	100.67	510.03	78 37
400	1.28	12.57	101.04	412.57	75 49
300	1.39	16.82	101.86	316.82	71 15
200	1.65	25.52	104.22	225.52	62 28
100	2.72	54.31	117.52	154.31	40 24

74. Thus let it be proposed to construct a bridge of 800 feet span, and let the adjunct weight of suspension rods, road-way, &c. be taken at one-half the weight of the chains: also let it be determined to load the chains with one-sixth of the breaking weight: to find the dimensions of the bridge.

The semi-span is 400 feet; hence the units of which y is 100 are 4 feet each; and l, the modulus of tenacity is $\frac{1}{4} \times 14800$ units, or 3700. Also m is $\frac{1}{8}$ and n is $\frac{1}{8}$. Hence $(1+m)(x+c)=nl$ gives $x + c = \frac{1}{9} l = 411.125$.

Now in the Table we find that the tension nearest in value to this is 412.57, which corresponds to

$$c = 400 \text{ and to } x = 12.57;$$

that is, in feet,

$$c = 1600, \qquad x = 50.$$

Also the angle which the chain makes with the horizon at the points of suspension is $90^0 - 75^0\,49'$, or $14^0\,11'$.

It would appear by such a Table that for a given span, the tension at the points of suspension is least when $x = \frac{1}{8}$ the whole span nearly.

In the preceding reasoning, the weight of any portion of a suspension bridge is supposed to be proportional to the corresponding length of the suspended chain. This however is not exact, and we shall now consider the question without introducing this supposition.

75. In the chain bridge, the strain proceeds from three causes; the weight of the suspended chain;—the weight of the road-way;—and the weight of the suspension rods which connect the former two together by means of vertical lines. The last of these weights will generally be small compared with the others.

In this case we shall still have as before, (fig. 118),

$$\frac{\text{tension at } C}{\text{weight of } CP} = \frac{dy}{dx}; \quad \text{whence weight of } CP = c\frac{dx}{dy},$$

including in the weight of CP, the three portions we have mentioned.

76. PROP. *To find the general equation to the curve of a suspension bridge.*

Let the weight of a unit of length of the curve be m.

Let the weight of a unit of length of the road-way be n; the road-way being supposed to be horizontal and of uniform weight for different equal portions of its length.

And let the weight of a unit of vertical surface of the suspending rods be r; the rods being supposed to be uniformly distributed, and very near each other; and therefore being reckoned as a vertical surface.

Let x, y, s be taken as before, Art. 46. If we take δy a small portion of the horizontal ordinate, and suppose δs to be the corresponding portion of the curve, $m\delta s$ is the weight of the portion of the curve, and $m\delta y$ the weight of the portion of the road-way. Also $rx\delta y$ is the weight of the corresponding portion of the rods. Hence the whole weight corresponding to the element δs is

$$m\delta s + n\delta y + rx\delta y$$

$$\text{or } \left\{m + n\frac{\delta y}{\delta s} + rn\frac{\delta y}{\delta s}\right\}\delta s.$$

When we suppose the curve to be continuous, we must suppose δs and δy to be indefinitely small; in which case the ratios of such quantities are the differential coefficients. Hence the differential coefficient of the weight is

$$m + n\frac{dy}{ds} + rx\frac{dy}{ds},$$

and the whole weight is the integral of this, taken with regard to s; that is, it is

$$ms + ny + r\int_s x\frac{dy}{ds};$$

the integral being supposed to begin when $x = 0$ and $y = 0$.

Hence the equation above stated becomes

$$ms + ny + r\int_s x \frac{dy}{ds} = c \frac{dx}{dy}.$$

The road-way is here supposed to be divided by trans verse cuts into indefinitely small separate parts, which i allowable, since the curve must always adjust itself so a to have the road-way a straight line.

77. Prop. *To find the nature of the curve when th weight of the suspension rods is neglected.*

In this case we make $r = 0$; and the equation is

$$ms + ny = c\frac{dx}{dy}; \text{ let } \frac{dx}{dy} = p.$$

$$ms + ny = cp;$$

$$c\frac{dp}{dy} = m\frac{ds}{dy} + n = m\sqrt{(1 + p^2)} + n;$$

$$\frac{cp\dfrac{dp}{dx}}{m\sqrt{(1 + p^2)} + n} = 1; \quad \text{assume } 1 + p^2 = q^2.$$

$$\frac{q\dfrac{dq}{dx}}{q + \dfrac{n}{m}} = \frac{m}{c}; \quad \text{whence } q - \frac{n}{m}\,\mathrm{l}\left(q + \frac{n}{m}\right) = \frac{mx}{c} + C;$$

$$\text{when } x = 0,\ p = 0,\ q = 1; \quad 1 - \frac{n}{m}\,\mathrm{l}\left(1 + \frac{n}{m}\right) = C.$$

$$q - 1 - \frac{n}{m}\,\mathrm{l}\frac{q + \dfrac{n}{m}}{1 + \dfrac{n}{m}} = \frac{mx}{c},$$

$$\text{or } \sqrt{(p^2+1)} - 1 - \frac{n}{m} \,\mathrm{l}\, \frac{\sqrt{(1+p^2)} + \dfrac{n}{m}}{1 + \dfrac{n}{m}} = \frac{mx}{c}.$$

Also

$$\frac{dy}{dx} = \frac{dy}{dp}\,\frac{dp}{dx} = \frac{c}{m\sqrt{(1+p^2)}+n}\cdot\frac{dp}{dx},$$

which may be rationalised by putting $\sqrt{(1+p^2)} = pq + 1$. We shall thus find

$$\frac{dy}{dq} = \frac{2c(1+q^2)}{(1-q^2)\{m+n-(n-m)q^2\}},$$

which may be integrated, because it is a rational fraction. The result will involve logarithms so long as n is greater than m.

If n be less than m, or the weight of a given length of road-way with its load be less than the weight of the same length of the chain, the logarithmic expression for y becomes imaginary, and the real integral will involve circular arcs. The reader will find this subject further discussed in the *Manchester Memoirs*, Vol. v. New Series; by Mr Hodgkinson, from whose paper the above investigation is taken.

CHAP. VI.

THE EQUILIBRIUM OF AN ELASTIC BODY.

78. Bodies are said to be *elastic* when they admit of a certain change of figure and dimensions, but possess a force which resists this change, which makes it depend upon the power applied, and which restores the bodies to their original dimensions and figure, when the power which altered them is removed. This restitutive energy acts in various ways.

1. *The Elasticity of Extension and Compression.*

A string may be stretched by a force applied lengthways to it, and an elastic surface or solid may be considered as a collection of elastic fibres.

It is found by experiment, that when a string is stretched, the increase of length is proportional to the force which produces it; that is, the *extension is as the tension**. We may also suppose the same law to extend to compression; but in order that a string may be susceptible of compression lengthways, it must be supposed to be inflexible.

2. *The Elasticity of Flexure.*

Wires and laminæ of different metals and other substances exert a force to unbend themselves when forcibly bent. In the flexure of elastic rods and laminæ, it appears by experiment that the deflexion, and consequently the curvature, is nearly as the force†. This also follows from

* See s'Gravesande's *Elem. Physices*, Lib. i. c. 26.

† See Boit's *Traité de Physique*, Tom. I. p. 509.

supposing an elastic rod to be composed of fibres which have elasticity of extension, as will be seen.

3. *The Elasticity of Torsion.*

Threads of metal, &c. when twisted, exert a force to untwist themselves. It appears from experiment*, that when very fine threads of metal are twisted by means of levers transverse to them, the force by which they tend to resume their natural state is very accurately as the angle of torsion.

1. *Elasticity of Extension.*

79. PROP. *When an elastic string of given length is stretched by a given force, to find its length.*

In a given elastic string the length added is, as we have said, proportional to the tension. If the tension be the same, the added length will, in different lengths of the same string, be proportional to the length; for it is manifest that a string two feet long will be twice as much extended by the same tension as a string one foot long; since the tension will be the same throughout, and therefore each of the halves of the first string will be as much stretched as the second string. In strings which differ in material, thickness, &c. the extension for a given length and tension, will be different for different substances; and will in each be proportional to a certain quantity which may be considered as the measure of the *extensibility* of the particular substance which is to be taken. If ϵ be this quantity for a certain string whose length at first (that is, when not stretched by any force) is a, when this string is stretched by a force or weight t, which will of course measure the tension, its increase of length will be proportional to $a\epsilon t$, and may be *equal* to this expression by properly assuming ϵ. Hence the length under these circumstances will be $a + a\epsilon t$, or $a(1 + \epsilon t)$.

---

* For the experiments of Coulomb, see Biot, *Traité de Physique*, Tom. I. p. 492.

We may determine ϵ if we know the original length of the string and its length for any given value of t. It may be convenient to know it in terms of the force which will draw out the string to *double* its length. Let E be this force; hence

$$a(1+\epsilon E)=2a; \quad \therefore \; \epsilon E=1, \text{ and } \epsilon=\frac{1}{E}.$$

Hence the length of the string under a tension t becomes

$$=a\left(1+\frac{t}{E}\right).$$

E may be expressed by a length of the given string whose weight would draw the string a to double its length. E is then called the *Modulus of Elasticity.*

If the tension be not the same throughout the string, this formula is not applicable. In this case we may suppose the string divided into indefinitely small portions; and in each of these portions the tension may be supposed constant, and the extension of that part found; and by combining all these, we get the extension of the whole.

80. Knowing thus the relation of the length and tension of such lines, we can easily express the conditions required by the solution of problems in which they occur, as will appear by the following examples.

PROB. I. *Fig.* 30. *AC, BC, are two given equal and similar elastic strings fixed at two points A, B, in the same horizontal line, and supporting at C a weight W: knowing the extensibility of the strings, to find where W will be supported; the strings themselves being supposed without weight.*

It is manifest that the vertical line CE will bisect AB. Let $AE=b$, angle $CAE=\alpha$, weight at $W=w$, tension of AC or $BC=t$, extensibility of $AC=\epsilon$, original length of $AC=a$, hence $AC=a(1+\epsilon t)$.

Since W is supported by the tensions of AC, BC, in those directions, we have

$$w = 2t \sin. \alpha; \quad \text{also } AE = AC . \cos. \alpha, \text{ or}$$

$$b = a(1 + \epsilon t) \cos. \alpha.$$

Eliminating t,

$$\frac{b}{a} = \cos. \alpha + \frac{\epsilon w}{2} \cot. \alpha \ldots\ldots\ldots (1).$$

If we should attempt to obtain α from this equation, we should arrive at an equation of four dimensions; and by solving this, we should find the position of equilibrium. But for the most common case, that is, when the extensibility is small, and the weight w not very large, we may easily deduce from our equation an approximation to the situation. For we have

$$\alpha = A + A'\epsilon + A'' . \frac{\epsilon^2}{1.2} + \ldots\ldots$$

when A, A', A'', ... are the values which α, $\frac{d\alpha}{d\epsilon}$, $\frac{d^2\alpha}{d\epsilon^2}$, ... assume by making $\epsilon = 0$, (Lacroix, *Elem. Treat.* Art. 21.)

Hence, putting 0 for ϵ in the fundamental equation, (1) and in its differentials, we obtain

$$\frac{b}{a} = \cos. A;$$

$$0 = -\sin. \alpha \frac{d\alpha}{d\epsilon} - \frac{\epsilon w}{2} \frac{1}{\sin.^2 \alpha} \frac{d\alpha}{d\epsilon} + \frac{w}{2} \cot. \alpha \frac{d\epsilon}{d\alpha};$$

$$\therefore A' = \frac{w}{2} . \frac{\cot. A}{\sin. A} = \frac{w}{2} . \frac{ab}{a^2 - b^2}, \text{ \&c.......}$$

Therefore

$$\alpha = A + \frac{w\epsilon}{2} . \frac{ab}{a^2 - b^2} + \text{\&c.......}$$

Here A is the angle BAC on the supposition that the strings

were inextensible: hence $\frac{w\epsilon}{2} \cdot \frac{ab}{a^2 - b^2}$ is, when ϵ is small, very nearly the quantity by which this angle is increased by supposing the strings extensible.

COR. 1. If $a = b$, that is, if the string ACB be just equal to AB when not stretched, we have from (1)

$$1 = \cos. \alpha + \frac{\epsilon w}{2} . \cot \alpha; \text{ and multiplying by } \tan. \alpha,$$

$$\tan. \alpha - \sin. \alpha = \frac{\epsilon w}{2}; \text{ and expanding } \tan. \alpha = \sin. \alpha (1 - \sin.^2 \alpha)^{-}$$

$$\frac{1}{2} \sin.^3 \alpha + \frac{1 . 3}{2 . 4} \sin.^5 \alpha + \ldots . = \frac{\epsilon w}{2}.$$

If ϵ be small, α will be small; hence, neglecting the higher powers of sin. α,

$$\sin.^3 \alpha = \epsilon w; \quad \sin. \alpha = \sqrt[3]{(\epsilon w)} = \sqrt[3]{\frac{w}{E}}.$$

E being the tension which would double the string. Hence for the same string, fixed horizontally and not stretched, the small deflexion produced by a weight hung at the middle point is as the cube root of the weight.

COR. 2. If $a < b$, the string would not reach from A to B horizontally without being stretched.

In this case, the equation becomes, multiplying by tan. α,

$$\frac{b}{a} \tan. \alpha - \sin. \alpha = \frac{\epsilon w}{2}.$$

And when α is small, neglecting its higher powers, we may put α both for its sine and tangent; hence

$$\frac{b-a}{a} . \alpha = \frac{\epsilon w}{2} = \frac{w}{2E}; \quad \alpha = \frac{w}{2E} \cdot \frac{a}{b-a}.$$

Therefore in this case the deflexion varies as the weight w, if it be supposed small.

81. PROB. II. *A uniform elastic string hangs vertically, stretched by its own weight: to find its length.*

Let ϵ, as before, be its extensibility when its weight is not supposed to act. Let a be its length when it is supposed not stretched; and x the distance, on the same supposition, of any small element δx from the upper extremity, by which it is suspended. The part below the element δx is $a - x$, when it is not stretched; and as the quantity of matter is not altered by extension, the weight of this part when stretched is as $a - x$; and may be represented by $a - x$, if we represent weights by the corresponding lengths of the unstretched string. Hence the element δx will become

$$\delta x \left\{1 + \epsilon (a - x)\right\};$$

or if z be the distance from the upper extremity to a point whose distance in the unstretched state was x,

$$\frac{dz}{dx} = 1 + \epsilon (a - x);$$

$$\therefore\ z = x - \frac{\epsilon (a - x)^2}{2} + \text{constant};$$

and at the upper extremity where $x = 0$, $z = 0$;

$$\therefore\ z = x + \frac{\epsilon (2ax - x^2)}{2}.$$

At the lower extremity, $x = a$; let the stretched length $= l$;

$$\therefore\ l = a + \frac{\epsilon a^2}{2}.$$

Hence, $\dfrac{\epsilon a^2}{2}$ is the quantity by which the length of the string is increased when it is hung up. If E be a length of the string whose weight alone would be sufficient to stretch any part to twice its length, $\epsilon = \dfrac{1}{E}$, and $\dfrac{a^2}{2E}$ is the increment of length.

Cor. 1. If we had $a = E$, we should have the length when stretched $= E + \dfrac{E^2}{2E} = \dfrac{3E}{2}$.

Cor. 2. Since $l = a\left(1 + \dfrac{\epsilon a}{2}\right)$; it appears that the weight of the string stretches it half as much as if it were all collected at the lowest point.

82. Prob. III. *To find the catenary when the chain is extensible.*

Let the chain or cord be of uniform thickness and density, and let, as before, the elasticity be such that a length a becomes $a(1 + \epsilon t)$ by a tension t.

Let C, fig. 123, be the lowest point; and let the tension at C be equal to the weight of a length $CA = c$ of the unstretched string: $AN = x$, $NP = y$ the horizontal and vertical co-ordinates: s = the arc CP, and s' = the length of CP before it was stretched, which may therefore represent the weight of CP; t = the tension at P.

If δs, $\delta s'$ be corresponding elements of s, s', we have

$$\delta s = \delta s'\,(1 + \epsilon)t; \quad \therefore \frac{ds'}{ds} = \frac{1}{1 + \epsilon t}.$$

The forces which keep CP at rest are the tension t at P, the tension c at C, and the weight s'. Hence these forces are as the sides of a triangle which are parallel to them; for instance, the elementary triangle at P, whose sides would be the elements δx, δy, δs: hence

$$\frac{t}{c} = \frac{ds}{dx}; \quad \frac{s'}{c} = \frac{dy}{dx};$$

By the second of these equations,

$$\frac{d^2y}{dx^2} = \frac{1}{c}\frac{ds'}{dx} = \frac{1}{c}\frac{ds}{dx}\frac{ds'}{ds} = \frac{\dfrac{ds}{dx}}{c + c\epsilon t};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{ds}{dx}}{c + c^2\epsilon\frac{ds}{dx}} \text{ by the first.}$$

If we make $\frac{dy}{dx} = p$, $\frac{ds}{dx} = \sqrt{(1+p^2)}$; and supposing the differentiation to be performed with respect to x, our last equation becomes

$$\frac{dp}{dx} = \frac{\sqrt{(1+p^2)}}{c + c^2\epsilon\sqrt{(1+p^2)}};$$

$$\therefore \frac{dx}{dp} = \frac{c}{\sqrt{(1+p^2)}} + c^2\epsilon;$$

$$\text{and } \frac{dy}{dp} = \frac{dy}{dx}\frac{dx}{dp} = p\frac{dx}{dp} = \frac{cp}{\sqrt{(1+p^2)}} + c^2\epsilon p.$$

Integrating these equations in p, we obtain

$$x = c\,\mathrm{l}\,\{p + \sqrt{(1+p^2)}\} + c^2\epsilon p;$$

$$y = c\sqrt{(1+p^2)} + \tfrac{1}{2}c^2\epsilon p^2;$$

the integrals being taken so that at C, where $p = 0$, we may have $x = 0$, and $y = c$.

By eliminating p, we should have the relation between x and y: and p is the tangent of the angle which the curve at P makes with the horizon.

$$\frac{ds}{dx} = \sqrt{(1+p^2)} = c\frac{dp}{dx} + c^2\epsilon\sqrt{(1+p^2)}\cdot\frac{dp}{dx};$$

$$\therefore s = cp + \tfrac{1}{2}c^2\epsilon\,\{p\sqrt{(1+p^2)} + \mathrm{l}\,[p + \sqrt{(1+p^2)}]\},$$

$$t = c\frac{ds}{dx} = c\sqrt{(1+p^2)}.$$

It appears that the values of x, y, and s, consist of two parts; namely, terms independent of ϵ, which are the same as they would be in a cord not extensible; and terms which

involve ϵ. Hence if CP and CP' be arcs of an extensible and of an inextensible catenary, for which the value of c, that is, the tension at C, is the same; and the values of p the same, that is, the tangents, PT, $P'T'$ parallel; $P'O$ and OP being horizontal and vertical, we have

$$P'O = c^2 \epsilon p, \quad OP = \tfrac{1}{2} c^2 \epsilon p^2.$$

The tension t is the same in both cases, and CP' is the length of CP not stretched.

Cor. If PT meet OP' in Q, $OQ = \dfrac{OP}{p} = \tfrac{1}{2} c^2 \epsilon p = \tfrac{1}{2} OP'$.

From these few examples it will be seen how problems involving extensible lines may be reduced to calculation.

2. *Elasticity and Resistance of Solid Materials.*

83. All solid substances, as wood, stone, metals, &c., are susceptible of some compression and extension. This compression and extension are greater as the forces producing them are greater; and when the forces produce a compression or extension greater than the texture of the substance can bear, the bodies are crushed or broken. We shall here find the change of figure of such bodies when they are compressed under given circumstances.

We shall suppose that all solid bodies may be considered as made up of elastic fibres, capable of extension and compression. We shall also suppose, as in the last Section, that the resistance to extension is proportional to the extension in each fibre, and the same of compression. We shall further assume, that the resistance to extension and to compression are the same in the same fibre.

These principles would follow if we were to suppose the particles of bodies to be kept in equilibrium by their mutual forces in the natural state of the body; and the change to be small, which they undergo by the action of any force. In this case it might be proved that the displacement of a

given particle would be ultimately as the force which produces it.

When a solid body is acted on by any force, it may be partly extended and partly compressed. Thus let a mass $ABQP$, fig. 124, be acted upon by a force F, compressing it in the direction EF. The surface PNQ may be brought into the direction pNq; in this case all the fibres RR' which are on one side of N are shortened; all those on the other side of N are lengthened. NN' remains the same as in the natural state. N is called the *neutral point*, and the line which separates the parts of a transverse section of the body which are compressed, from those which are elongated is called the *neutral line* of that section.

84. PROP. *When a rectangular prismatic mass is compressed by a force parallel to the direction of the axis; to find the neutral line.*

Let AB, fig. 124, be the rectangular base of the mass, MM' its axis. And let the slice $UTPQ$ be compressed so as to assume the form $UTpq$, N being the neutral line. Then any fibre parallel to the axis, as VR, is compressed so that its length becomes Vr: and by the supposition, if t be the force compressing it, E the modulus of elasticity, as in last Article; we shall have

$$Rr = VR\frac{t}{E}; \quad \text{and hence } t = E\frac{Rr}{VR}.$$

Let $PM = MQ = a$, $MF = h$, $MR = x$, and the breadth of the beam perpendicular to $AB = b$; $MN = n$, whence $RN = n + x$; force at $F = f$.

Also let UT and QP meet in O, and let $OK = \rho$.

Hence

$$\frac{Rr}{VR} = \frac{Rr}{NL} = \frac{NR}{OL} = \frac{n+x}{n+\rho}.$$

And the force of VR, supposing its breadth and thickness each 1, is

$$t = E \cdot \frac{Rr}{VR} = E \cdot \frac{n+x}{n+\rho}.$$

Hence if we take a very thin portion, of which the thickness is δx and breadth b, its force is

$$E \cdot \frac{n+x}{n+\rho} \cdot b\delta x,$$

and this is the increment of the force exerted at R corresponding to δx. When x is negative and greater than n, this is negative; and accordingly the compression for that part becomes extension.

The forces which keep each other in equilibrium are the force f acting at F, and the elementary forces of all the fibres VR. And hence, by Art. 24, we must have, 1st, the force f equal to all the forces

$$E \cdot \frac{n+x}{n+\rho} b\delta x;$$

and 2nd, the moment of the force f about N equal to the moments of all the forces

$$E \cdot \frac{n+x}{n+\rho} b\delta x \text{ about } N.$$

Also the aggregate of all the forces will be found by taking the coefficients of δx, in the expressions so found, and the integrals of these differential coefficients from

$$x = -a, \text{ to } x = a.$$

Hence we have

$$f = \int_x E \frac{n+x}{\rho+n},$$

$$f(h+n) = \int_x E \frac{(n+x)^2}{\rho+n}.$$

Integrating between the proper limits,

$$f = E \cdot \frac{2nab}{\rho+n},$$

AD. $AX = x$, $XM = y$. And since the curve is nearly straight line, $\frac{dy}{dx}$ is small: hence the radius of curvature

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ is } = \frac{1}{\frac{d^2y}{dx^2}}, \text{ nearly.}$$

But by Cor. 2. to last Art. if $AD = l$, $k = DX = l - x$,

$$\text{rad. of curv.} = \frac{E}{F} \cdot \frac{a^2}{3(l - x)};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{F}{E} \cdot \frac{3(l - x)}{a^2}.$$

Integrate with respect to x, observing that $\frac{dy}{dx} = 0$ hen $x = 0$,

$$\therefore \frac{dy}{dx} = \frac{F}{E} \cdot \frac{3lx - \frac{3}{2}x^2}{a^2}.$$

Integrate again: observing that $y = 0$, when $x = 0$.

$$y = \frac{F}{E} \cdot \frac{\frac{3}{2}lx^2 - \frac{1}{2}x^3}{a^2}.$$

And if the whole deflexion $DE = \delta$, making $x = l$,

$$\delta = \frac{F}{E} \cdot \frac{l^3}{a^3}.$$

Cor. 1. If we put for F its value $\frac{f}{2ab}$, we have

$$\delta = \frac{fl^3}{2Ea^3b}.$$

Hence it appears that for a given breadth and thickness the deflexion is as the force and cube of the length.

And for a given force and length the deflexion is inversely as the breadth and cube of the thickness.

COR. 2. Let the direction of the tangent at E make an angle θ with the tangent at A. Then θ may be called the *angular deflexion.*

And $\frac{dy}{dx} = \tan.\theta$; hence, putting l for x in the value of $\frac{dy}{dx}$, at the extremity, $\tan.\theta = \frac{F}{E} \cdot \frac{3l^2}{2a^2} = \frac{3fl^2}{4a^3b}$.

The extreme angular deflexion is as the force and square of the length.

87. PROP. *When a rectangular prismatic beam, fixed in a horizontal position, is bent by its own weight, (its thickness being vertical) to find the deflexion.*

In Art. 85, Cor. 2; put Fk the moment of the force which bends the beam $= (l-x)\frac{l-x}{2} = \frac{1}{2}(l-x)^2$; and for the rad. of curv. $\dfrac{1}{\frac{d^2y}{dx^2}}$.

Hence we have

$$\frac{d^2y}{dx^2} = \frac{3(l-x)^2}{2Ea^2}; \quad \frac{dy}{dx} = \frac{l^3-(l-x)^3}{2Ea^2};$$

$$y = \frac{l^3x + \frac{1}{4}(l-x)^4 - \frac{1}{4}l^4}{2Ea^2};$$

and the whole deflexion $\delta = \frac{3l^4}{8Ea^4}$.

COR. In this and the last Article, δ being observed, E may be found.

88. PROP. *When an isosceles triangular prism is acted upon by any force in any direction, to find the neutral point at any part.*

The force is supposed to act in the plane which bisects the vertical angle of the isosceles triangle. Let $ABQP$, fig. 124, be this plane, the vertex of the triangle being at P, and its base at Q.

Let $OT=\rho$, $TV=x$, $TL=n$, $TU=a$, $PF=h$, $MFy=a$, and the force $=f$, modulus of elasticity $=E$.

As before, in Art. 85, we shall have the force of a single fibre at $R = E\dfrac{NR}{OL} = E\dfrac{n-x}{\rho+n}$.

And whatever be the form of the section perpendicular to the plane $ABQP$, if y be the ordinate of this section perpendicular to the line PQ, we shall have for the elementary force exerted at R,

$$E\frac{n-x}{\rho+n}y\,\delta x.$$

And by the same reasoning as in Art. 85,

$$f\sin. a = E\int_x \frac{n-x}{\rho-n}y,$$

$$f(h+n)\sin. a = E\int_x \frac{(n-x)^2}{\rho+n}y.$$

In the case of the triangle, $y = mx$, m being a constant quantity. And integrating from $x=0$ to $x=a$,

$$f\sin. a = \frac{Em}{\rho+n}\cdot\left(\frac{1}{2}na^2 - \frac{1}{3}a^3\right),$$

$$f(h+n)\sin. a = \frac{Em}{\rho+n}\left(\frac{1}{2}n^2a^2 - \frac{2}{3}na^3 + \frac{a^4}{4}\right);$$

$$\therefore\ h + n = \frac{6n^2 - 8na + 3a^2}{6n - 4a}$$

$$= \frac{3a^2 - 4an}{6n - 4a} + n;$$

$$\therefore h = \frac{3a^2 - 4an}{6n - 4a};$$

$$\therefore n = \frac{3a^2 + 4ah}{4a + 6h}.$$

COR. 1. If $h = 0$, or the force act at P, $n = \frac{3}{4}a$.

COR. 2. If the force act perpendicularly to the prism, h is infinite, and $n = \frac{2a}{3}$.

COR. 3. If the force act above P, h will be negative. Thus if the force act at Q, $h = -a$, $n = \frac{a}{2}$.

COR. 4. To find the radius of curvature of the neutral line, we have

$$\text{rad. curv.} = \rho + n = \frac{Em}{f\,\text{sin.}\,a}\left(\frac{1}{2}na^2 - \frac{1}{3}a^3\right);$$

and putting for n its value,

$$\text{rad. curv.} = \frac{Em}{f\,\text{sin.}\,a} \cdot \frac{a^4}{6(4a + 6h)} = \frac{Ema^4}{36f\left(h + \frac{2a}{3}\right)\text{sin.}\,a}.$$

And if we take a point distant from P by $\frac{2}{3}$ PQ, and from this point draw a perpendicular on the line of direction of the force; if this perpendicular $= k$,

$$k = \left(h + \frac{2a}{3}\right)\text{sin.}\,a;\quad \text{rad. curv.} = \rho + n = \frac{Ema^4}{36fk};$$

or if b be the base of the triangle, $ma = b$, $\rho + n = \dfrac{Ea^3b}{36fk}$.

COR. 5. If f be the weight of a length F of the prism, $= \frac{1}{2}Fab$;

$$\therefore\ \rho + n = \frac{Ea^2}{18Fk}.$$

In the same manner we might find the neutral point for rismatic beams of other figures. And the deflexion when ıey are acted on by given weights would be found in the ıme manner as before.

Also if the beams are not prismatic, a will be variable; nd by putting for it the expression belonging to each case, 'e may find the deflexion in beams of other forms.

89. PROP. *A rectangular prismatic beam is compressed y a given force acting in a direction parallel to the axis; find the deflexion.*

Let $ABA'B'$, fig. 126, be the beam, FF' the line in which ıe force acts. P any point in the axis. And since the eflexion is supposed to be small, PM, which is perpendiular to FF', may be considered as perpendicular also to he axis. Hence if a be half the thickness of the beam $= \frac{1}{2}AB$) and n the distance of the neutral point above P, $CM = x$, $PM = y$, we have, by Art. 85, $n = \dfrac{a^2}{3y}$.

Also if ρ be the radius of curvature of the axis CP, by Cor. 2, of the same Article,

$$\rho + n = \frac{E}{F} \cdot \frac{a^2}{3y}; \quad \therefore\ \rho = \left\{\frac{E}{F} - 1\right\} \frac{a^2}{3y} = \frac{c^2}{y}, \text{ suppose.}$$

Now $\dfrac{1}{\rho} = -\dfrac{d^2y}{dx^2}$ nearly, because the deflexion is small;

$$\therefore\ \frac{d^2y}{dx^2} = -\frac{y}{c^2}.$$

$$\text{Integrate, } \therefore\ \frac{dy^2}{dx^2} = C - \frac{y^2}{c^2}.$$

And if k be EV, the greatest ordinate, $y = k$ when $\frac{dy}{dx} = 0$;

$$\therefore \frac{dy^2}{dx^2} = \frac{k^2 - y^2}{c^2}; \quad \frac{1}{\sqrt{(k^2-y^2)}} = -\frac{1}{c}\frac{dx}{dy};$$

$$\therefore \text{arc}\left(\cos. = \frac{y}{k}\right) = \frac{x}{c}; \ x \text{ being measured from } E,$$

$$y = k \cos. \frac{x}{c}.$$

Let $l = EF$ = half the length of the beam. And let $h = CF$, the distance of the force from the axis. Therefore when $x = l$, $y = h$,

$$h = k \cos. \frac{l}{c}; \quad y = h \cdot \frac{\cos. \frac{x}{c}}{\cos. \frac{l}{c}}.$$

Hence $EV = h \sec. \frac{l}{c}$; and DV the deflexion $= EV - FC$;

$$\therefore \text{deflexion} = h\left\{\sec. \frac{l}{c} - 1\right\}.$$

$$\text{But } c^2 = \frac{a^2}{3}\left\{\frac{E}{F} - 1\right\}; \quad \therefore \frac{l}{c} = \frac{l}{a}\frac{\sqrt{3F}}{\sqrt{E-F}}.$$

Cor. 1. If E be very large compared with F, we shall have the deflexion

$$= h\left\{\sec. \frac{l\sqrt{3F}}{a\sqrt{E}} - 1\right\}.$$

Cor. 2. The radius of curvature at V

$$= \frac{c^2}{k} = \frac{c^2 \cos. \frac{l}{c}}{h} = \frac{a^2 \cos. \frac{l}{c}}{3h}\left\{\frac{E}{F} - 1\right\}.$$

And when E is very large compared with F,

$$\text{rad. curv. at } V = \frac{Ea^2}{3Fh} \cos \frac{l\sqrt{3F}}{a\sqrt{E}}.$$

COR. 3. The deflexion will be greater, as the secant, in Cor. 1, is greater; and when the secant is infinite, the formula will fail; in this case the prism will either be crushed, or will bend so much that the above reasoning is no longer applicable. And this will be the case if the arc be a quadrant. Hence in order that the prism may support a weight with a small deflexion, the weight acting on one side of the axis, we must have

$$\frac{l\sqrt{3F}}{a\sqrt{E}} < \frac{\pi}{2},$$

$$\frac{l^2}{a^2} < \frac{\pi^2 E}{12F}.$$

COR. 4. If the force act at the extremities of the axis, $h = 0$; and there will be no deviation except the secant of the arc be infinite; that is, except

$$\frac{l^2}{a^2} = \frac{\pi^2 E}{12F} = .8225 \frac{E}{F}.$$

Hence we may find the weights which columns of given materials will support. Thus, if in fir-wood the modulus E be 10000000 feet, a bar an inch square and 10 feet long may begin to bend when

$$F = .8225 \times \left(\frac{1}{120}\right)^2 \times 10000000 = 571 \text{ feet};$$

that is, it will bend when pressed by the weight of 571 feet of the same bar, or about 120 pounds, neglecting the pressure arising from the weight of the bar itself.

The modulus of elasticity for iron or steel is about 9000000 feet; for wood, from 4000000 to 10000000; and for stone, probably about 5000000.

Cor. 5. In the same manner we might find the deflexion of a triangular prismatic beam acted on by a longitudinal force. For in this case, supposing E large with respect to F,

$$\rho = \frac{Ea^2}{18Fy}.$$

3. *The Curves formed by Elastic Laminæ.*

90. If we consider the thickness of the elastic bodies in Art. 85, to be small, we may neglect n, and we have, when the section of the body is a rectangle,

$$\rho = \frac{2Ea^3b}{3fk};$$

and in all cases $\rho = \frac{\mathrm{E}}{fk}$; when E is a constant quantity depending upon the size and form of the section of the elastic body, and upon its elasticity. If we suppose the body to be a lamina of uniform thickness, the value of a will be constant, and E will be proportional to b.

Prop. 91. *An elastic lamina of uniform breadth and thickness is fixed at one end and acted upon by a given force; it is required to determine the form of the curve.*

Let BA, fig. 127, be the lamina, fixed at B; f the force, which acts at A or E in the direction AE; $CM = x$, $MP = y$, co-ordinates perpendicular and parallel to the direction of the force AE; $AP = s$. The radius of curvature at P is

$$-\frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Now it will manifestly make no alteration in the curvature at any point, as P, whether, after the equilibrium is established,

we suppose the part PA rigid or not, or of one form or another. Hence the force f may be supposed to act on a straight rigid arm $PK = x$; and we have by the last Article,

$$fk = \frac{\text{E}}{\rho}, \text{ or } fx = -\frac{\text{E}\dfrac{d^2y}{dx^2}}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}} \dots\dots\dots (1).$$

If $\dfrac{dy}{dx} = p$, this becomes

$$fx = -\frac{\text{E}\dfrac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}};$$

and integrating,

$$\frac{f}{2}(b^2 + x^2) = -\frac{\text{E}p}{\sqrt{(1+p^2)}} \dots\dots\dots (2),$$

b^2 being an arbitrary constant, to be determined. Hence, obtaining p^2,

$$p^2 = \frac{f^2(b^2+x^2)^2}{4\text{E}^2 - f^2(b^2+x^2)^2}; \text{ and, making } a^2 = \frac{2\text{E}}{f},$$

$$p^2 = \frac{(b^2+x^2)^2}{a^4 - (b^2+x^2)^2} = \frac{(b^2+x^2)^2}{(a^2-b^2-x^2)(a^2+b^2+x^2)};$$

$$\therefore \frac{dy}{dx} = \pm \frac{b^2+x^2}{\sqrt{(a^2-b^2-x^2)(a^2+b^2+x^2)}};$$

$$\text{also } \frac{ds}{dx} = \sqrt{(1+p^2)} = \pm \frac{a^2}{\sqrt{(a^2-b^2-x^2)(a^2+b^2+x^2)}}.$$

We must determine b^2 from known circumstances in the problem. If the curve BP be continued, to meet the line AE, and at the point of intersection make with the line of abscissas an angle α, we shall easily determine b^2. Since at that point $x = 0$ and $p = \tan.\, \alpha$, equation (2) becomes

$$\frac{fb^2}{2} = -\frac{\text{E} \tan.\, \alpha}{\sec.\, \alpha}; \quad \therefore b^2 = -\frac{2\text{E} \sin.\, \alpha}{f} = -a^2 \sin.\, \alpha.$$

If the curve do not meet the line AE, b^2 must be otherwise determined, as will be seen hereafter.

92. Making $a^2 - b^2 = c^2$, whence $a^2 + b^2 = 2a^2 - c^2$, our equations become

$$\frac{dy}{dx} = \pm \frac{a^2 - c^2 + x^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \text{.........} (3).$$

$$\frac{ds}{dx} = \pm \frac{a^2}{\sqrt{(c^2 - x^2)(2a^2 - c^4 + x^2)}} \text{.........} (4).$$

When $x = 0$, as at A, the curve makes an angle α with the abscissa. When $a^2 - c^2 + x^2 = 0$, or $x^2 = c^2 - a^2 = -b^2 = a^2 \sin. \alpha$, or $x = a \sin.^{\frac{1}{2}} \alpha$, we have $\dfrac{dy}{dx} = 0$ and the curve is parallel to the abscissa.

When $x = c$, $\dfrac{dy}{dx}$ becomes infinite, and the curve is perpendicular to the axis. When x is greater than this, the expression is impossible. Hence $c = ED$. Beyond this point the curve turns back with an arc DC, fig. 127, similar to the arc AD before this point: and these two arcs correspond to the double sign of $\dfrac{dy}{dx}$ in (3).

If we find the radius of curvature we shall obtain it $= \dfrac{a^2}{2x}$. Hence the radius of curvature at the points A, C, A', &c. where $x = 0$, is infinite. These are points of contrary flexure, and the curve between each successive two of them consists of similar arcs placed alternately. The curve, as determined from the equation, may be continued indefinitely in this form.

93. To obtain the values of y and s we should have to integrate equations (3) and (4). The expressions, however,

cannot be integrated in finite terms*. We may easily integrate them in series, by making $\sqrt{(c^2-x^2)}=u$; whence we have, neglecting the signs,

$$\frac{ds}{dx}=\frac{a^2}{u\sqrt{(2a^2-u^2)}},\quad \frac{dy}{dx}=\frac{a^2-u^2}{u\sqrt{(2a^2-u^2)}},$$

$$\frac{ds}{dx}-\frac{dy}{dx}=\frac{u}{\sqrt{(2a^2-u^2)}}.$$

Expanding $\dfrac{1}{\sqrt{(2a^2-u^2)}}$ by the binomial theorem, these equations become

$$\frac{ds}{dx}=\frac{1}{\sqrt{2}}\cdot\left\{\frac{a}{u}+\frac{1}{4}\cdot\frac{u}{a}+\frac{1\,.\,3}{4\,.\,8}\,\frac{u^3}{a^3}+\&\text{c.}\right\},$$

$$\frac{ds}{dx}-\frac{dy}{dx}=\frac{1}{\sqrt{2}}\cdot\left\{\frac{u}{a}+\frac{1}{4}\cdot\frac{u^3}{a^3}+\frac{1\,.\,3}{4\,.\,8}\cdot\frac{u^5}{a^5}+\&\text{c.}\right\}.$$

It is only necessary to take the integrals from $x=0$, to $x=c$, which give AD and ED, fig. 127. Now, since

$$u=\sqrt{(c^2-x^2)},$$

---

* These expressions are of the kind which have been called *Elliptical Transcendentals*, from their connexion with the functions on which the rectification of elliptical arcs depends. Though the integration cannot be effected rigorously, many properties and relations of them have been discovered, and methods of finding the integrals within any requisite degree of approximation. The student will find these very completely treated of in the *Exercices de Calcul Integral* of Legendre; to whom, along with Euler and Lagrange, we are indebted for the discoveries made in this province of analysis.

If we make $x=c\,.\sin.\,\phi$, we shall find $s=\pm\frac{a}{\sqrt{2}}\cdot\int_\phi\frac{1}{\sqrt{(1-m^2\,.\sin.^2\phi)}}$, putting $\frac{c^2}{2a^2}=m^2$: which is what Legendre calls an elliptical function of the first order, and designates by F. Similarly, y is reducible to elliptical functions. It appears from the work above-mentioned, that though we cannot find the length of an arc s, we can determine arcs double, treble, &c. or the halves, thirds, &c. of given arcs; with many other properties, for which the reader is referred to the work itself. We can also obtain very converging series for the integrals; both when m is small, (which we have given in the text,) and when m is nearly $=1$; and likewise for other cases, in which the calculation is facilitated by the Tables given by Legendre.

we shall have, between these limits,

$$\int_x \frac{1}{u} = \int_x \frac{1}{\sqrt{(c^2 - x^2)}} = \frac{\pi}{2};$$

and by the known methods of finding $\int_x (c^2 - x^2)^{\frac{2n+1}{2}}$, (Lacroix's *Elem. Treat.* Art. 171.), we shall find between the same limits

$$\int_x u = \frac{1}{2} \cdot \frac{\pi}{2} \cdot c^2;$$

$$\int_x u^3 = \frac{1.3}{2.4} \cdot \frac{\pi}{2} \cdot c^4; \text{ and so on.}$$

Hence if the length $ADC = l$, and the height $AC = h$;

$$\frac{l}{2} = \frac{\pi a}{2\sqrt{2}} \cdot \left\{1 + \frac{1^2}{2^2} \cdot \frac{c^2}{2a^2} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{c^4}{4a^4} + \dots\right\}.$$

$$\frac{l-h}{2} = \frac{\pi a}{2\sqrt{2}} \cdot \left\{1 - \frac{1.c^2}{2a^2} + \frac{1^2.3}{2^2.4} \cdot \frac{c^4}{2a^4} + \frac{1^2.3^2.5}{2^2.4^2.6} \cdot \frac{c^6}{4a^6} + \dots\right\};$$

$$\therefore\ l = \frac{\pi a}{\sqrt{2}} \cdot \left\{1 + \frac{1^2}{2^2} \cdot \frac{c^2}{2a^2} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{c^4}{4a^4} + \dots\right\} \dots\dots\dots (5)$$

$$h = \frac{\pi a}{\sqrt{2}} \cdot \left\{1 - \frac{1^2}{2^2} \cdot \frac{3}{1} \cdot \frac{c^2}{2a^2} - \frac{1^2.3^2}{2^2.4^2} \cdot \frac{5}{3} \cdot \frac{c^4}{4a^4} - \dots\right\} \dots (6)$$

and knowing

$$\frac{a}{\sqrt{2}} = \sqrt{\frac{\text{E}}{f}}, \text{ and } \frac{c^2}{2a^2} = \frac{a^2 - b^2}{2a^2} = \frac{1 + \sin\,\alpha}{2} = \cos.^2\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\alpha\right),$$

we may calculate l and h approximately.

From equation (3) we must determine the species of the curve. They will depend on the value of c compared with a.

94. Prop. *When the elasticity is variable, to determine the curve, having given the elasticity, and conversely.*

We have supposed the moments of the forces which tend to bend the lamina at any point to be equal to $\frac{E}{\rho}$, where E is the measure of the elasticity, and is the same for every point. We shall now suppose E to be a function of the curve or its co-ordinates. As before, let a force f act on the lamina and let the abscissa be perpendicular to the direction of the force. Hence

$$fx = \frac{E}{\rho}; \quad \therefore E = fx\rho.$$

If E be given in terms of s, or of x and y, we may substitute and integrate. If E be to be found, it will be had from the formula,

$$E = -\frac{fx\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

E, as appears from Art. 90, may be supposed proportional to the breadth, when the thickness is constant, and to the cube of the thickness, when the breadth is constant.

PROB. *To find how the breadth of a uniform elastic lamina must vary, that by a weight hung at the end of it, it may be bent into the form of a quadrant. Fig.* 128.

In this case ρ is constant; therefore E, $=f\rho x$, is as x. Hence the lamina must be such that its projection ACc on a horizontal plane is a triangle.

95. Hitherto we have supposed that the elastic rod or lamina in its natural state, when it is not acted on by any forces, is a straight line. But we may suppose that it is naturally of any form whatever, and that it is deflected from this natural form by the same laws by which we before supposed it deflected from a straight line.

PROP. *In an elastic rod which is naturally a given curve, the curvature produced by any force at any point*

is equal to the natural curvature, together with the curvature which the same force would produce in a rectilinear rod of the same elasticity, acting in the same manner.

Let Pq, fig. 129, be a small given arc whose natural curvature is Pq, and its center of curvature o; and let it be bent into the position PQ, with its center of curvature at O, by means of a force acting at the arm QE. Then the deflexion Qq, of Q from its natural position, is the same which it would be if Pq were a straight line.

Now ultimately, when PQ or Pq is indefinitely small, Qq may be considered as perpendicular to the tangent at P, and will therefore be equal to the difference of the perpendiculars QR and qr upon the tangent. Hence (Newt. *Prin.* Lem. XI.)

$$Qq = QR - qr = \frac{PQ^2}{2PO} - \frac{Pq^2}{2Po} = \frac{PQ^2}{2} \cdot \left\{\frac{1}{PO} - \frac{1}{Po}\right\}.$$

But if Pq were a straight line, Po would be infinite; and if $Q'q'$ be the deflexion in this case for an arc PQ', and PO' the radius of curvature for the same force;

$$Q'q' = \frac{PQ'^2}{2} \cdot \frac{1}{PO'}.$$

And by supposition the deflexion from the natural form is the same in the two cases for the same arc: or $Q'q' = Qq$, PQ' being equal to PQ. Hence

$$\frac{1}{PO} - \frac{1}{Po} = \frac{1}{PO'},$$

$$\text{and } \frac{1}{PO} = \frac{1}{PO'} + \frac{1}{Po};$$

and the curvature being inversely as the radius, the Proposition is manifest.

COR. Since, by Art. 90, $\dfrac{1}{PO'} = \dfrac{\text{E}}{fk}$,

we have $\dfrac{1}{PO} = \dfrac{fk}{\text{E}} + \dfrac{1}{Po}$.

E being, as before, a quantity which measures the elasticity of the rod PA; and fk the moment of the force which acts.

96. Prop. *A uniform elastic rod, which is naturally a given curve, is fixed at one end and acted on by a given force: it is required to find the form which it will assume.*

Let BA, fig. 127, be the curve when the force f is applied. And as before, CM perpendicular to $AE = x$, $MP = y$, $AP = s$. And let the radius of curvature of any point P be, in the original form, $= r$, and in the form which it assumes, $= \rho$. Hence

$$\frac{1}{\rho} = \frac{fx}{\text{E}} + \frac{1}{r}, \text{ or } fx = \frac{\text{E}}{\rho} - \frac{\text{E}}{r};$$

and r being given in terms of s, we have a differential equation to the curve AB.

97. Prob. *Fig.* 130. *The curve Ba, being originally a quadrant, fixed at its lowest point B, it is required to find the curve BA, when it is acted on by the force F.*

Let FA meet the horizontal line BD in D: $DM = x'$, $MP = y$; radius of $Ba = r$; and since the original curvature is in a direction contrary to that which the force would produce, r must be made negative in the formula. Hence it becomes

$$fx' = \frac{\text{E}}{\rho} + \frac{\text{E}}{r}; \text{ or if we make } x' - \frac{\text{E}}{fr} = x,$$

$$fx = \frac{\text{E}}{\rho}; \text{ and, putting for } \rho \text{ its value,}$$

$$fx = -\frac{\text{E}\dfrac{d^2y}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}};$$

which agrees with equation (1), Art. 91, for the common elastic curve.

Hence the curves into which the circular rod can be bent are the same as those which may occur in the case of the straight lamina.

If we take $DE = \frac{\text{E}}{fr}$, we shall have $EM = x$, and hence if EC, perpendicular to DE, meet the curve, it will cut it in a point of contrary flexure C.

98. Prob. *Fig.* 131. *To find what must be the natural form of a lamina aB, that a force F, acting perpendicularly at its extremity, may deflect it into a straight line AB.*

For the same reason as before r must be negative. Also ρ is infinite. And if a point p be, by the action of the force, brought to P, we have $AP = ap = s$, suppose; hence,

$$fs = \frac{\text{E}}{r}, \text{ or } rs = \frac{\text{E}}{f}; \text{ or, making } \frac{2\text{E}}{f} = a^2,$$

$$rs = \frac{a^2}{2};$$

which equation contains the property of the curve.

If pn be perpendicular on an, and if we make $an = x$, $np = y$, and the angle $ptn = \phi$, we shall have

$$\frac{dx}{ds} = \cos. \phi, \quad \frac{dy}{ds} = \sin. \phi, \quad r = \frac{ds}{d\phi}.$$

Hence $\frac{d\phi}{ds} = \frac{1}{r} = \frac{2s}{a^2}$; $\phi = \frac{s^2}{a^2}$, the arbitrary constant being $= 0$ if an be a tangent at a.

$$\therefore \frac{dx}{ds} = \cos. \frac{s^2}{a^2}, \quad \frac{dy}{ds} = \sin. \frac{s^2}{a^2};$$

and by integrating these expressions, we should have the values of x and y in terms of s.

We may integrate by expanding $\cos. \frac{s^2}{a^2}$ and $\sin. \frac{s^2}{a^2}$, and thus we obtain

$$x = s - \frac{s^5}{1.2.5a^4} + \frac{s^9}{1.2.3.4.9a^8} - \&c.$$

$$y = \frac{s^3}{1.3a^2} - \frac{s^7}{1.2.3.7a^6} + \frac{s^{11}}{1.2.3.4.5.11.a^{10}} - \&c.$$

which converge rapidly, except when s is very large in comparison with a.

Since the curvature increases in proportion to the distance from a, it is manifest that the curve will be a kind of spiral, which will tend to a point C with an infinite number of revolutions. The co-ordinates of this point C would be found, if we could find the values of x and y when s is infinite, which cannot be obtained from the series given above.

It makes no difference what point of the spiral we take for the point B. If we suppose that point and its tangent to be fixed, the portion of the curve Ba may always be bent into a straight line.

4. *Elasticity of Torsion.*

99. When a slender thread of metal, &c. is twisted, it tends to resume its natural condition, and would communicate angular motion to any body to which it is annexed, for instance, to a straight rod or rigid line fastened across it at right angles. A force acting on this rod may resist this tendency to motion, and produce equilibrium. The force necessary for this purpose is, as has been already mentioned, proportional to the angle through which the thread is twisted. Let there be a thread, perpendicular at C, fig. 132, to the plane of the paper. Let its upper extremity be fixed, and let Bb be a bar suspended at its lower extremity in a horizontal position. If this needle be turned out of the position Bb in which it would naturally hang, into any other Pp, the force which, acting at P in a horizontal plane and per-

pendicular to CP, would retain it in this position, will be as the arc BP, or as the angle BCP. If BC vary, the equilibrium will be preserved so long as the product of the force ($=F$) and distance BC remains the same; hence

$$F \, . \, BC \propto BCP.$$

If we call the angle BCP, θ, and the distance $CB = CP$, a, we shall have $Fa \propto \theta$, and $Fa = \epsilon\theta$, by properly assuming ϵ. The quantity ϵ is manifestly the value of the force F when the arm $BC = 1$, and the angle $\theta = 1$; it is different for different substances and masses, and may be considered as measuring the *elasticity of torsion.*

Problems in which elasticity of torsion enters present few difficulties; especially as there is no change of figure in the bodies which are concerned. We shall therefore only give one instance of their solution.

100. PROB. *Fig.* 132. *The extremity P of the bar whose natural position is Bb, is acted on by a repulsive force which varies inversely as the square of the distance from the center of force A, and is kept in its place by torsion; given its position, to find the force at A.*

Let the force of repulsion exerted by A be $\frac{f}{z^2}$; z being the distance AP. This force acts in the direction AP. Let it be resolved into two, one in the direction MP, of the lever CP, and the other in TP, perpendicular to CP. The former of these produces no effect to turn the lever CP, and the latter only is balanced by the torsion.

Let $ACP = \theta$, and $APT = ApP = \frac{1}{2} ACP = \frac{1}{2}\theta$.

Hence the force which balances the torsion is $\frac{f}{z^2} \cos. \frac{1}{2}\theta$.

Let $CA = a$, and we have manifestly $z = AP = 2a \sin. \frac{1}{2}\theta$.

Hence the force which balances the torsion is $\frac{f \cos. \frac{1}{2}\theta}{4a^2 \sin.^2 \frac{1}{2}\theta}$.

Let now $ACB = \beta$, and the angle BCP, to which the torsion is proportional, will be $\theta + \beta$. The force of torsion will be $\epsilon(\theta + \beta)$ acting at P, perpendicular to CP; as is stated in last Article. Hence

$$\frac{f \cos. \frac{1}{2}\theta}{4a^2 \sin.^2 \frac{1}{2}\theta} = \epsilon(\theta + \beta);$$

whence

$$f = 4a^2 \epsilon(\theta + \beta) \sin. \tfrac{1}{2}\theta \tan. \tfrac{1}{2}\theta.$$

If θ correspond to another position of Pp, f' being the force which retains it there, we have

$$f' = 4a^2 \epsilon(\theta' + \beta) \sin. \tfrac{1}{2}\theta' \tan. \tfrac{1}{2}\theta',$$

whence

$$\frac{f}{f'} = \frac{(\theta + \beta) \sin. \frac{1}{2}\theta \tan. \frac{1}{2}\theta}{(\theta' + \beta) \sin. \frac{1}{2}\theta' \tan. \frac{1}{2}\theta'}.$$

If the arcs θ, θ' be very small, we may put the arc for its sine and tangent; and hence

$$\frac{f}{f'} = \frac{(\theta + \beta)\theta^2}{(\theta' + \beta)\theta'^2}.$$

If the points B and A coincide, $\beta = 0$,

$$\frac{f}{f'} = \frac{\theta \sin. \frac{1}{2}\theta \tan. \frac{1}{2}\theta}{\theta' \sin. \frac{1}{2}\theta' \tan. \frac{1}{2}\theta'};$$

and when θ, θ' are small,

$$\frac{f}{f'} = \frac{\theta^3}{\theta'^3}.$$

The combination supposed in this proposition agrees with the *Torsion Balance* of Coulomb, which has been employed for the purpose of measuring very small repulsive and attractive forces. In some cases the instrument was constructed with so much delicacy, that each degree of torsion required a force of only $\frac{1}{122400}$ of a grain.

CHAP. VII.

ON THE STRENGTH OF MATERIALS.

101. WHEN solid bodies are made to undergo flexure beyond a certain degree, they break or undergo fracture: we now proceed to consider the force which is requisite, in order to produce this effect.

When bodies break, the fracture may *begin* either by the *tearing* of the extended parts, or by the *crushing* of the compressed parts. Thus if a beam be supported on two props, and broken by a weight hung to its middle point, the first destruction of the original texture of the beam may be either a *crack* on the under side of the beam, or a piece crushed or started out, on the opposite side.

As soon as the fracture is begun, it will have a tendency to extend across the beam; for the flexure will become greater and the power of resisting less by the failing of one part; therefore the remainder of the section of the beam will give way either by tearing or crushing.

We assume that the line which separates the parts torn and the part crushed, in a fractured section, will be the line which separated the parts extended and compressed, the instant before fracture.

102. In treating of the elasticity of materials, (Chap. VI.) it has been supposed that the resistance of the material to extension and to compression are on the same scale or modulus. In Art. 84, in the formula $t = E\frac{Rr}{VR}$, t represents the force

her of extension or compression, and E is the same in e two cases. But in reality the resistance of most mate-ls to extension and to compression is different; whether consider the resistance to flexure or to fracture.

Moreover it has been supposed that the resistance to ension and to compression in each fibre is proportional the extension and to the compression; and the truth of s supposition can only be known by reference to expe-ient.

The former supposition, of the equality of the modulus extension and compression, requires to be corrected, in ler that we may apply our conclusions to practice with nmon materials, as wood and cast iron. It appears, for h substances, that in the case of moderate flexures the dulus of compression is generally less than the modulus of ension; and for the forces which are called into play en beams are broken, it appears that the difference of se moduli is still more considerable: as will appear in the sequent articles.

The assumption that the forces of extension and com-ssion vary each respectively as the extension and com-ssion, appears to be more nearly true. For moderate cures it was proved to be true by Mr Hodgkinson (*Man-ster Memoirs*, Vol. IV. new series). He bent beams in such nners that in some cases they were capable of extension y, and in others of compression only; and he proved that both cases, the deflexion produced was as the force applied nsversely; whence it followed (Art. 86.) that the forces are the extension, and as the compression respectively.

In the case of flexures produced in breaking, it follows, m Mr Barlow's experiments, that the same law is very arly true, as will be shewn in subsequent articles.

103. PROP. *The modulus of elasticity being different for mpression and extension, to find the position of the utral line in any beam exposed to a transverse strain.*

Let the beam be fixed perpendicularly in a wall, and bent by a force F, acting at an arm l, in a direction parallel to the transverse section. The sum of the forces of compression and the sum of the forces of extension, which act on the two sides of the neutral line, must be equal to each other; for the transverse force which bends the beam, is parallel to the transverse section, and cannot balance any portion of either of these forces.

Let δa be any portion of the area of the extended transverse section, and x the distance of this portion from the neutral line, then the extension will be as x. Also let E be the force of extension of a fibre at a distance h from the neutral line, and let $\phi(x)$ be the function of the extension to which the force is proportional. Then the force exerted by the area δa will be

$$\frac{E\phi(x)}{\phi(h)}\delta a,$$

and the sum of all these forces is the whole force of extension.

In like manner if C be the force of compression of a fibre at the distance h from the neutral line, we shall have a similar expression for the force of compression at any point. Hence equating these expressions:

$$E \times \text{sum of all the } \phi(x) \,.\, \delta a = C \times \text{sum of all the } \phi(x) \,.\, \delta a.$$

The sum on the first side being taken for the extended, and on the second for the compressed area.

Cor. 1. If the force of the fibres be the same for all degrees of extension and compression; (Galileo's hypothesis;)

$$\phi(x) = 1,$$

$$E \times \text{extended area} = C \times \text{compressed area}.$$

Cor. 2. If the forces of extension and compression be proportional to the extension and compression

$$E \times \text{sum of all the } x\,.\,\delta a = C \times \text{sum of all the } x\,.\,\delta a.$$

Therefore by the property of the center of gravity

$$E \times \text{extended area} \times e = C \times \text{compressed area} \times c:$$

e and c being the distances of the centers of gravity of the extended and compressed areas respectively from the neutral line.

Hence in this case

$$\frac{\text{extended area}}{\text{compressed area}} = \frac{Cc}{Ee}.$$

Cor. 3. In a rectangular beam bent transversely, the center of gravity of each area is in the middle of its length. Hence, on the supposition of Cor. 2,

$$\frac{2e}{2c} = \frac{Cc}{Ee} \text{ and } \frac{C}{E} = \frac{e^2}{c^2} \text{ and } C : E :: e^2 : c^2.$$

In moderate strains the forces of extension and compression are nearly as the extension and compression. Hence this corollary is here applicable.

104. It appeared by Mr Hodgkinson's experiments, that in rectangular beams of fir, exposed to moderate flexure, the depths of the section extended and compressed were in the ratio of 169 to 190 nearly. Hence

$$E : C :: (190)^2 : (169)^2 :: 100 : 79 :: 5 : 4 \text{ nearly.}$$

In Mr Barlow's experiments fir beams were broken; and it then appeared that the areas extended and compressed, were as 3 to 5 nearly. Hence in this case

$$E : C :: 25 : 9 :: 11 : 4 \text{ nearly;}$$

if the forces be in this case as the extensions and compressions, which it will hereafter appear they are, nearly.

The ratios from these different experiments are very different. It is possible, that in the very act of breaking a considerable change takes place in the proportion of the extended and compressed areas.

105. PROP. *The same suppositions being made as in the last Proposition, to find the position of the neutral line, in a beam, the section of which is an isosceles triangle, and which is bent in a plane perpendicular to the base of the triangle; the vertex of the triangle being in the extended side of the beam.*

We shall assume $\phi(x) = x$ as in Cor. 2, of last Proposition.

Resume the equation of last Proposition,

$$E \times \text{sum of all the } \phi(x) \,.\, \delta a = C \times \text{sum of all the } \phi(x) \,.\, \delta a.$$

If b be the base of the triangle, g and h the height of the compressed and extended portions respectively from the neutral line; we have, for the extended surface,

$$\delta a = \frac{b}{g+h}(h - x)\,\delta x, \text{ and since } \phi(x) = x,$$

The sum of all the $\phi(x) \,.\, \delta a$

$$\text{is } \frac{b}{g+h}\left(\tfrac{1}{2}hx^2 - \tfrac{1}{3}x^3\right) \text{ from } x = 0, \text{ to } x = h;$$

$$\text{that is, it is } \frac{b}{g+h} \cdot \frac{h^3}{6}.$$

For the compressed surface,

$$\delta a = \frac{b}{g+h}(h + x)\,\delta x,$$

and the sum of all the $\phi(x) \,.\, \delta a$ is

$$\frac{b}{g+h}\left(\tfrac{1}{2}hx^2 + \tfrac{1}{3}x^3\right), \text{ from } x = 0 \text{ to } x = g;$$

$$\text{that is, it is } \frac{b}{g+h}\left(\tfrac{1}{2}hg^2 + \tfrac{1}{3}g^3\right);$$

Hence

$$Eh^3 = C(3hg^2 + 2g^3);\quad \frac{E}{C}\cdot\quad \frac{g^2}{h^2} + 2\frac{g^3}{h^3}.$$

Hence $\frac{E}{C}$ being known, $\frac{g}{h}$ may be found.

Cor. 1. The determination of the ratio $\frac{g}{h}$ from the above equation would require the solution of a cubic equation. The relation of $\frac{g}{h}$ and $\frac{E}{C}$ may be determined more simply, by means of the following Table:

$h = 12$	$C = 1728$	$C = 1$
$g = 12$	$E = 8640$	$E = 5.00$
11	7018	4.06
10	5600	3.24
9	4374	2.53
8	3328	1.92
7	2450	1.42
6	1728	1.00
5	1150	.66
4	704	.40
3	378	.22
2	160	.09
1	38	.02

It appeared in an experiment of Mr Barlow on an equilateral prism of fir, (*On the Strength of Timber*, 3d ed. p. 172) that g was .75 of an inch, and h was .982 of an inch.

T

Hence in this case $\frac{g}{h}$ was $\frac{9.17}{12}$, and by the Table, $\frac{E}{C}$ was a little greater than 2.5; and, as appears by interpolation, was 2.64 nearly.

106. PROP. *The same suppositions being made as in the last proposition, and the edge being on the compressed side, to find the position of the neutral line.*

The reasoning will be nearly the same as before. If h' and g' be the height of the compressed and extended portions from the neutral line, and E, C the force of extension and compression at the distance h' from that line,

$$E\,(3h'g'^2 + 2g'^3) = Ch'^3;$$

$$\frac{E}{C} = 3\frac{g'^2}{h'^2} + 2\frac{g'^3}{h'^3}.$$

COR. 1. Hence the same Table as before, (Cor. 1, of last Prop.) will give the relation between $\frac{h'}{g'}$ and $\frac{C}{E}$, putting in the table

$$\frac{g'}{h'} \text{ for } \frac{g}{h} \text{ and } \frac{C}{E} \text{ for } \frac{E}{C}.$$

In an experiment of Mr Barlow on an equilateral prism of fir it appeared (p. 173) that g' was .39 of an inch, and h' was 1.342 inch. Hence in this case $\frac{h'}{g'}$ was $\frac{3.5}{12}$, and by the Table, $\frac{C}{E}$ was .31 nearly. This gives $\frac{E}{C} = 3.22$, which is not much different from the value obtained in the last Article, namely, 2.64.

If we had $g : h :: 5 : 6$ when the edge is extended, and $g' : h' :: 7 : 24$ when the edge is compressed, we should obtain from both cases nearly the same ratio of the force of extension to that of compression, namely, $E : C :: 3.2 : 1$ *nearly*.

The approximate coincidence shews that the supposition on which the above investigations proceed, namely, that the forces of extension and compression in each fibre are as the extension and compression, is nearly true up to the limit of fracture; supposing the neutral line, as shewn by the fracture, to be the neutral line before the fracture.

107. PROP. *Having given the position of the neutral line, and the absolute force of direct cohesion of the material, to find the force which applied transversely to any beam will break it.*

Let F be the force, and l the arm at which it is applied perpendicularly. The force F and the forces of extension and compression must balance each other about the neutral axis. Hence their moments must be equal, and we have, retaining the notation of Article 103, and using the abbreviation Σ to express "the sum of all the"

$$Fl = \Sigma E \frac{\phi(x)}{\phi(h)} x \delta a + \Sigma C \frac{\phi(x)}{\phi(h)} x \delta a.$$

Whence

$$Fl = \frac{E}{\phi(h)} \Sigma \phi(x) . a \delta a + \frac{C}{\phi(h)} \Sigma \phi(x) . x \delta a \ldots (2),$$

which is to be combined with the equation of Article 103;

$$E \Sigma \phi(x) \delta a = C \Sigma \phi(x) \delta a \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In general we may eliminate C by (1), and thence find the relation of E and F by (2).

In the case of fracture, $E \delta a$ is the force which will break a fibre, having a section δa, at the distance h from the neutral line. Therefore E is the force of direct cohesion for a surface 1, and is therefore known by proper experiments.

108. PROP. *Having given the position of the neutral axis in a rectangular beam, and the force necessary to break it: to find the law of the force of extension.*

Let the beam, fixed in a wall, be broken by the tearing the extended surface, by means of a force F acting transrsely at the extremity of the beam, the length of the ›am being l. This force must, at the moment of fracture, alance the forces of extension and compression. Hence

$$Fl = \frac{E}{\phi(h)} \Sigma \phi(x) . x \delta a + \frac{C}{\phi(h)} \Sigma \phi(x) . x \delta a.$$

And in this case, h is the distance from the neutral line to the point of greatest extension, at which fracture begins, and E is the greatest force of extension which the substance can exert; or that with which it just yields to direct divulsion.

If we suppose $\phi(x)$ to be x^m, and the beam to be rectangular, the equation of the former proposition, Article 108, namely,

$$E \Sigma \phi(x) \delta a = C \Sigma (\phi x) \delta a,$$

will give us, using integration to find the sums,

$$E h^{m+1} = C g^{m+1};$$

g and h being the whole length of the sections of compression and extension measured from the neutral line.

Also on the same supposition, b being the breadth of the section, the equation (2) of last Article gives

$$Fl = E \times \frac{b h^{m+2}}{(m+2) h^m} + C \times \frac{b g^{m+2}}{(m+2) h^m};$$

whence, by the previous equation, we find

$$Fl = \frac{E b h^{m+2} + E b h^{m+1} g}{(m+2) h^m} = \frac{E b h}{m+2} (h+g);$$

$$m + 2 = \frac{E b h (h+g)}{Fl}.$$

Ex. It appeared by Mr Barlow's experiments (p. 168) that a fir beam 24 inches long and 2 inches square, fixed at one end in a wall, required a weight of 558 lbs. at its extremity to produce fracture. The neutral point or axis was at about $\frac{3}{8}$ the depth of the beam. The force of direct cohesion on a square inch of the same wood was 13000 lbs.

Therefore in inches $b = 2$, $h + g = 2$, $h = \frac{3}{8}$ of $2 = \frac{3}{4}$, $l = 24$; also $F = 558$ and $E = 13000$: hence

$$m + 2 = \frac{13000 \times \frac{3}{4} \times 2}{558 \times 24} = 3 \times \frac{13000}{13392}.$$

This is very nearly 3; therefore $m + 2 = 3$ and $m = 1$ nearly.

Thus the assumption of the previous Articles that the force of extension is as the extension is confirmed.

109. Prop. *The same suppositions being made as in the previous propositions, to find the force which, acting transversely as in Art.* 108, *will break a beam the section of which is an isosceles triangle; the edge being extended.*

Retaining the notation of the preceding Articles 105 and 106, we have equation (2) as before.

$$Fl = \frac{E}{h}\Sigma x^2 \delta a + \frac{C}{h}\Sigma x^2 \delta a.$$

Now for the extended area,

$$\delta a = \frac{b}{g+h}(h - x)\delta x.$$

Hence

$$\Sigma x^2 \delta a = \frac{b}{g+h}\left\{\frac{hx^3}{3} - \frac{x^4}{4}\right\} = \frac{b}{g+h}\cdot\frac{h^4}{12}$$

And for the compressed area,

$$\delta a = \frac{b}{g+h}(h + x)\delta x.$$

Hence

$$\Sigma x^2 \delta a = \frac{b}{g+h}\left\{\frac{hx^3}{3} + \frac{x^4}{4}\right\} = \frac{b}{g+h} \cdot \frac{4hg^3 + 3g^4}{12}.$$

Whence equation (2) becomes

$$Fl = \frac{b}{12h(g+h)}\{Eh^4 + C(4hg^3 + 3g^4)\}.$$

But by Art. 105,

$$C = \frac{Eh^3}{3hg^2 + 2g^3}.$$

Hence

$$Fl = \frac{Ebh^2}{12(g+h)}\left\{h + \frac{4h+3g}{3h+2g}g\right\}$$

$$= \frac{Ebh^2}{12(g+h)} \frac{3(g+h)^2}{3h+2g} = \frac{Ebh^2(g+h)}{4(3h+2g)},$$

which gives the value of F for the case of fracture.

Cor. 1. In nearly the same manner we shall find for the case in which the edge is compressed, the beam being broken by extension,

$$F'l = \frac{Ebg'^2(g'+h')}{4h'}.$$

Cor. 2. Hence the strength when the edge is extended is to the strength when the edge is compressed,

$$F : F' :: \frac{h^2(g+h)}{3h+2g} : \frac{g'^2(g'+h')}{h'}.$$

Let $g + h = g' + h' = c$, the depth of the beam. Therefore $3h + 2g = 2c + h$.

$$F : F' :: \frac{h^2}{2c+h} : \frac{(c-h')^2}{h'}.$$

In the cases referred to in Articles 105 and 106, it appeared that when $E : C :: 3.2 : 1$ nearly,

$$h \text{ was nearly } \frac{5}{11} c \text{ and } h' \text{ nearly } \frac{24}{31} c.$$

Hence

$$F : F' :: \frac{25}{27 \times 11} : \frac{49}{31 \times 24} :: 18600 : 14553 :: 372 : 291.$$

The forces which produced fracture in these two cases were found by experiment to be 370 and 313 pounds respectively (*Barlow*, pp. 172, 173).

When fracture is produced by compression we may in like manner determine the force of a beam, knowing the force which is to resist a given surface of the material acting directly.

110. When a uniform beam rests on two props and is pressed by a weight in the middle, the effect is the same as if it were fixed at the middle and acted on by a transverse force at its extremity equal to the pressure on each of the props, that is, to half the weight in the middle.

Prop. *When a weight is supported on any point of a beam resting horizontally on two props, the requisite strength of the beam at each point is as the rectangle of the segments of the length of the beam.*

Let a beam rest on two props, the length between the props being l: and let a weight W rest on a point of the beam, the distances of which point from the props are p and q. The pressures on the two props are respectively

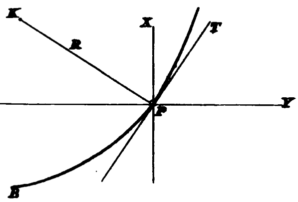
$$\frac{Wp}{l} \text{ and } \frac{Wq}{l};$$

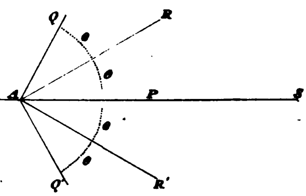
and the moment of each of these to turn the correspondin
end of the beam round the point where W rests is

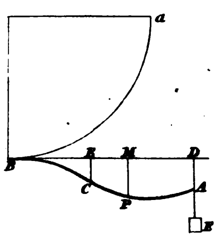
$$\frac{Wpq}{l}.$$

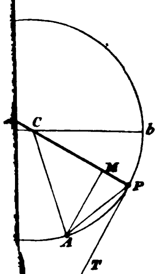
Hence the strength of the beam at different points must l
as pq, the rectangle of the segments of the beam.

97









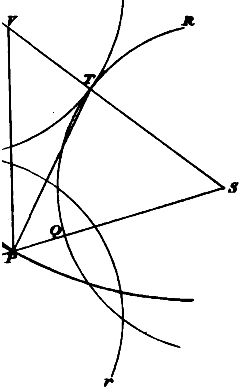
[illegible] Burleigh St. Strand.

1.

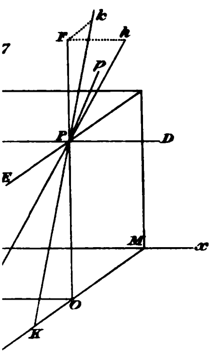
al
el

H
as

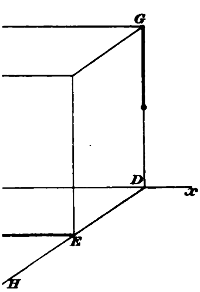
102



7



108



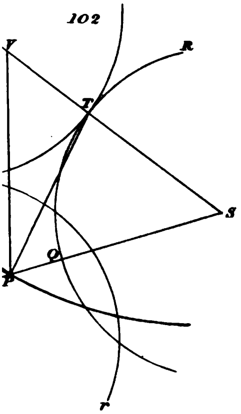
J.&J. Neele sc. 352 Strand.

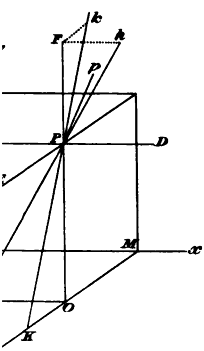
1.

a
e

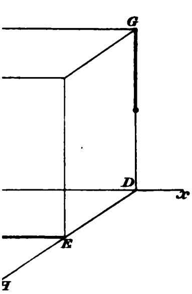
H
a

102

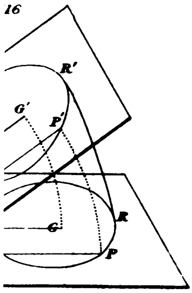


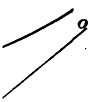


'08

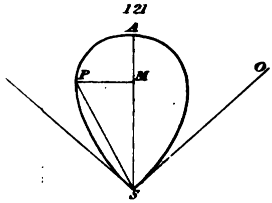


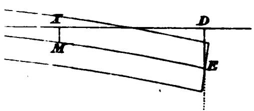
16

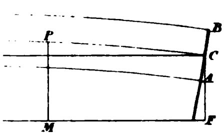




121

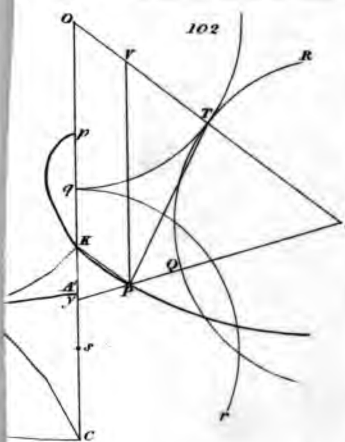




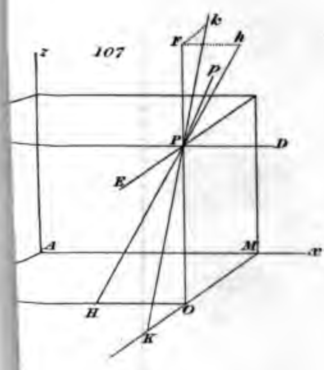


Neale & Cº sc. Burleigh St. Strand.

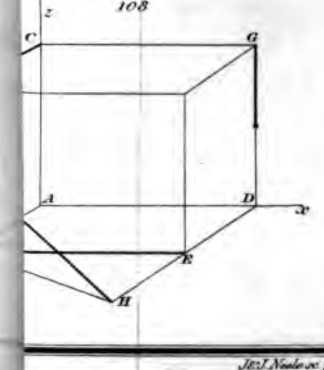
102



107



108



J&I. Neale sc. 351 Strand